Colloquium Biometricum 42
2012, 5–15

PROBLEMS OF MISSING OBSERVATIONS IN LINEAR MODELS IN THE WORKS OF PROFESSOR WIKTOR OKTABA

Andrzej Kornacki, Jacek Wawrzosek

Department of Applied Mathematics and Computer Science
University of Life Sciences in Lublin
Akademicka 13, 20-950 Lublin, Poland
e-mail: andrzej.kornacki@up.lublin.pl; jacek.wawrzosek@up.lublin.pl

This paper is dedicated to the memory of Professor Wiktor Oktaba.

Summary

The purpose of this paper is to present some of the scientific achievements of Professor Wiktor Oktaba on inference in the Gauss-Markov and Zyskind-Martin models with missing observations and the transformation of these models. The Professor's achievements include his works from the 1980s. The first section contains key information and symbols in the Gauss-Markov and Zyskind-Martin models, taking into consideration missing observations. The next paragraph briefly presents the main research results on the prediction of the vector of missing observations in the general Gauss-Markov and Zyskind-Martin models. The third paragraph presents the results of the comparison of the actual model (the model with available observations) to the complete model (the model supplemented by predictors of missing observations). The following paragraph describes the results related to the properties of the predictors of missing observations. The last (fifth) paragraph presents the results of the Professor's work on the invariant linearly sufficient transformations of the Gauss-Markov model.

Keywords and phrases: Gauss-Markov model, Zyskind-Martin model, missing observations

Classification AMS 2010: 62B05, 62F03, 62J99

1. Introduction and basic models and symbols

In the final period of Professor Wiktor Oktaba's professional activity before he decided to enjoy his fully deserved retirement we had the privilege to get acquainted with the research topics in the scope of his and his colleagues' interest and to carry out the research alongside the Professor. The Professor presented us tables which demonstrated amazingly clearly some issues in mathematical statistics which are still part of unexplored areas of modern linear models. As young researchers working under his experienced eye it was our task to generalize some results, published in the subject literature. Our collaboration with the Professor began in this way. His experience, orderliness and perseverance in the pursuit of his goals resulted in our finding of solutions to some of the problems, which gave us an opportunity to learn about his working techniques as well as his kindness and firmness.

For the purpose of presenting the scientific achievements of Professor Oktaba this chapter will deal with an introduction of basic symbols and concept of models crucial for further discussion. The ***general Gauss-Markov model*** is understood as a linear model:

$$\mathbf{y} = \mathbf{X}\beta + \mathbf{e}\,, \qquad (1.1)$$

where y is a random vector of the n observation with the expected value $\mathbf{X\beta}$ and covariance matrix $\sigma^2\mathbf{V}$. It is assumed here that the matrix X (with n □ p dimensions) is known, the vector of parameter $\boldsymbol{\beta}$ is unknown, σ^2 signifies an unknown scalar, and $\mathbf{V}$ any singular or non-singular known matrix. Model (1.1) is symbolised by the $(\mathbf{y}, \mathbf{X\beta}, \sigma^2\mathbf{V})$ and denoted by GM.

According to the unified theory of the least squares method (Rao 1973) in the GM model the special matrix T as $\mathbf{T} = \mathbf{V} + \mathbf{XBX}'$ is analysed, where $\mathbf{B} = \mathbf{B}'$ is such that the following condition is satisfied:

$$R(\mathbf{V} \vdots \mathbf{X}) = R(\mathbf{T})\,, \qquad (1.2)$$

where $R(\mathbf{A})$ denotes the linear space spanned by the columns of a matrix A. This equality is equivalent to the set of conditions:

$$\begin{cases} r(\mathbf{V} \vdots \mathbf{X}) = r(\mathbf{T}) \\ R(\mathbf{X}) \subset R(\mathbf{T}) \quad \wedge \quad R(\mathbf{V}) \subset R(\mathbf{T}) \end{cases},$$

where r(**A**) denotes the rank of matrix **A**.

Moreover, if in the GM model the following condition is satisfied

$$R(\mathbf{X}) \subset R(\mathbf{V}), \tag{1.3}$$

then this GM model is called the ***Zyskind*-Martin *model*** and is denoted by ZM. In the literature on the subject also ZM is called a model with trivial deterministic part or a *weakly singular linear model*.

The GM and ZM models are a generalization of the standard model (where $\mathbf{V} = \mathbf{I}$ is the identity matrix)) and of the Aitken model (where $\mathbf{V}$ is a non-singular matrix; Aitken, 1934, Oktaba, 1982,a,b). In the cited works of the Professor, some examples illustrating the reasons for the appearance of singularity of covariance matrix $\mathbf{V}$ can be found.

Any matrix $\mathbf{A}^-$ satisfying the condition $\mathbf{AA}^-\mathbf{A} = \mathbf{A}$ is called a g-inverse of the matrix **A**. The matrix $\mathbf{A}^+$ satisfying the following four condition: $\mathbf{AA}^+\mathbf{A} = \mathbf{A}$, $\mathbf{A}^+\mathbf{AA}^+ = \mathbf{A}^+$, $(\mathbf{AA}^+)' = \mathbf{AA}^+$, $(\mathbf{A}^+\mathbf{A})' = \mathbf{A}^+\mathbf{A}$ is unique and is called the Moore-Penrose inverse matrix of *A*.

ZM model with the (1.3) condition appeared in the Zyskind and Martin paper (1969) in which the authors were the first who had generalized the Gauss-Markov theorem to include the case of the model with singular covariance matrix. They noted that in the GM model with some of the constraints concerning estimable functions, solutions to the set of equations $\mathbf{X'V^+X\beta} = \mathrm{X'V^+}\mathbf{y}$ do not lead to BLUE's of estimable functions. They also proved that in the GM model in the class of all *g*-inverses of the dispersion matrix **V** there is a nonempty subclass ϑ such that for any $\mathbf{V}^* \in \vartheta$ a solution $\hat{\boldsymbol{\beta}}^*$ of normal equations

$$\mathbf{X'V^*X\beta} = \mathbf{X'V^*y} \tag{1.4}$$

leads to the BLUE of any estimable linear function $\boldsymbol{\lambda}'\boldsymbol{\beta}$, which takes the form $\boldsymbol{\lambda}'\hat{\boldsymbol{\beta}}^*$. Moreover, every *g*-inverses $\mathbf{V}^-$ (also $\mathbf{V}^+$) belongs to ϑ if and only if condition (1.3) is fulfilled (Zyskind, Martin 1969, Oktaba 1982b).

The form (1.4) is simple and similar to that in the Aitken model ($\mathbf{X'V^{-1}X\beta} = \mathbf{X'V^{-1}y}$, where $\mathbf{V}$ is a non-singular matrix) and to that in the standard model ($\mathbf{X'X\beta} = \mathbf{X'y}$, where $\mathbf{V} = \mathbf{I}$) and also to that in the GM model (in the general case $\mathbf{X'T^-X\beta} = \mathbf{X'T^-y}$, where $\mathbf{V}$ is a singular matrix, $\mathbf{T} = \mathbf{V} + \mathbf{XBX'}$ and the condition (1.2) is fulfilled; Rao 1973).

Also, for testable linear hypotheses $\mathbf{L\beta}=\boldsymbol{\varphi}_0$, in order for the quadratic form for the hypothesis included in the numerator of the fraction in the test function F to be as in the standard model the difference of the sum of squares for errors:

$$\left(\mathbf{L}\hat{\boldsymbol{\beta}}-\boldsymbol{\varphi}_{\mathbf{0}}\right)'\left(\mathbf{L}\left((\mathbf{X}'\mathbf{M}\mathbf{X})^{-}-\mathbf{B}\right)\mathbf{L}'\right)^{-}\left(\mathbf{L}\hat{\boldsymbol{\beta}}-\boldsymbol{\varphi}_{\mathbf{0}}\right)= \\ =\min_{\mathbf{L\beta}=\boldsymbol{\varphi}_{\mathbf{0}}}(\mathbf{y}-\mathbf{X\beta})'\mathbf{M}(\mathbf{y}-\mathbf{X\beta})-\min_{\boldsymbol{\beta}}(\mathbf{y}-X\boldsymbol{\beta})'\mathbf{M}(\mathbf{y}-\mathbf{X\beta}) \tag{1.5}$$

for given **M**, the assumption (1.3) is necessary and sufficient (i.e., ZM) for $\mathbf{M}=\mathbf{V}^{-}$ (Rao 1972, 1973). Note that the equation (1.5) is a generalization of the Pythagorean theorem.

2. Prediction of missing observations in the general Gauss-Markov model

It is held that an error or loss of data means that the orthogonal model of the planned number of observations becomes a model with missing observations. Therefore in the GM model a partitioned vector y as $\mathbf{y}=\begin{bmatrix}\mathbf{y}_1\\ \mathbf{y}_2\end{bmatrix}$ is introduced, where $\mathbf{y}_1$ is the vector of m missing observations and $\mathbf{y}_2$ is the vector of (n-m) available observations. This assumption on the observation vector y does not reduce the generality of considerations. This follows directly from the properties for these transformations discussed in Chapter 5, which can be a permutation of the observations. The model:

$$(\mathbf{y}_2,\mathbf{X}_2\boldsymbol{\beta},\sigma^2\mathbf{V}_2) \tag{2.1}$$

containing exclusively the observations available is then called ***actual model*** and denoted by GM_a

Fisher (1960) proved that when $Cov(\mathbf{y})=\sigma^2\mathbf{I}$ the best linear predictor of missing observations is the predictor that minimises the sum of squares for error: $SS_{\mathbf{e},\mathbf{I}}=(\mathbf{y}-\mathbf{X}\hat{\boldsymbol{\beta}})'(\mathbf{y}-\mathbf{X}\hat{\boldsymbol{\beta}})$ to the vector of missing observations $\mathbf{y}_1$, where $\hat{\boldsymbol{\beta}}$ is the estimator of the parameter $\boldsymbol{\beta}$ obtained from normal equations $\mathbf{X}'\mathbf{X\beta}=\mathbf{X}'\mathbf{y}$. The Professor using the unified theory of the least squares method generalized this result on the Aitken and ZM models as well as on the general Gauss-Markov model GM (Oktaba, Jagiełło 1982, Oktaba et al. 1983, Oktaba et al. 1986). The main result is presented below:

Theorem 2.1

In the GM model (in the form of $(\mathbf{y}, \mathbf{X\beta}, \sigma^2\mathbf{V})$, where $\mathbf{V}$ is any singular or non-singular known matrix, for matrices $\mathbf{T} = \mathbf{V} + \mathbf{XBX}'$ and $\mathbf{B} = \mathbf{B}'$ the condition (1.2) is fulfilled and $\hat{\boldsymbol{\beta}}$ is the solution of normal equations $\mathbf{X}'\mathbf{T}^{-}\mathbf{X}\boldsymbol{\beta} = \mathbf{X}'\mathbf{T}^{-}\mathbf{y}$), the ***predictor*** *of the* ***vector*** *of* ***missing observations*** obtained by minimising the sum of squares for the error $SS_{\mathbf{e},\mathbf{T}^-} = (\mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}})'\mathbf{T}^{-}(\mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}})$ to the vector $\mathbf{y}_1$ is as follows:

$$\hat{\mathbf{y}}_1 = -(\mathbf{E}_1 + \mathbf{E}_1')^{-}(\mathbf{E}_2 + \mathbf{E}_3')\mathbf{y}_2 = \mathbf{Z}\mathbf{y}_2 \tag{2.2}$$

with the following conditions:

$$\begin{cases} \begin{bmatrix} \hat{\mathbf{y}}_1 \\ \mathbf{y}_2 \end{bmatrix} \in R(\mathbf{T}) \\ \quad (\mathbf{E}_2 + \mathbf{E}_3')\mathbf{y}_2 \in R(\mathbf{E}_1 + \mathbf{E}_1'), \\ \quad \text{where} \\ \mathbf{T}^{-}(\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{T}^{-}\mathbf{X})^{-}\mathbf{X}'\mathbf{T}^{-}) = \begin{bmatrix} \mathbf{E}_1 & \mathbf{E}_2 \\ \mathbf{E}_3 & \mathbf{E}_4 \end{bmatrix} \end{cases} \tag{2.3}$$

and the dimensions of the matrices $\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3, \mathbf{E}_4$ are respectively *m*×*m*, *m*×(*n*-*m*), (*n*-*m*)×*m*, (*n*-*m*)× (*n*-*m*).

Theorem 2.1 is a generalization of the results of Yates (1933), Fisher (1960), Oktaba and Jagiełło (1982) for standard $(\mathbf{V} = \mathbf{I})$ and non-standard Aitken's $(|\mathbf{V}| \neq 0)$ models.

3. Actual and complete models

As well as the actual model, the ***complete model*** is also considered, where instead of the vector missing observations its predictor (2.2) is used. In view of (2.2) the complemented model is as follows:

$$\left(\begin{bmatrix}\hat{\mathbf{y}}_1\\ \mathbf{y}_2\end{bmatrix}=\begin{bmatrix}\mathbf{Z}\\ \mathbf{I}\end{bmatrix}\mathbf{y}_2,\ \begin{bmatrix}\mathbf{Z}\\ \mathbf{I}\end{bmatrix}\mathbf{X}_2\boldsymbol{\beta},\ \sigma^2\begin{bmatrix}\mathbf{Z}\\ \mathbf{I}\end{bmatrix}\mathbf{V}_2\left[\mathbf{Z}'\mathbf{I}\right]\right) \tag{3.1}$$

and is denoted by GM_c (or ZM_c respectively). The Professor Oktaba in his works (Oktaba et al. 1983, Oktaba et al. 1986) proved the following results:

Theorem 3.1

a) If the actual model (2.1) is a ZM model then the complete model (3.1) is also ZM;

b) Quadratic forms:

$$\begin{cases} SS_{\mathbf{y}_2}=(\mathbf{y}_2-\mathbf{X}_2\boldsymbol{\beta})'((\mathbf{y}_2-\mathbf{X}_2\boldsymbol{\beta})) \quad \text{and} \\ SS_{[\hat{y}_1',y_2']'}=\left(\begin{bmatrix}\mathbf{Z}\\ \mathbf{I}\end{bmatrix}\mathbf{y}_2-\begin{bmatrix}\mathbf{Z}\\ \mathbf{I}\end{bmatrix}X_2\beta\right)'\left(\begin{bmatrix}\mathbf{Z}\\ \mathbf{I}\end{bmatrix}\mathbf{V}_2\left[\mathbf{Z}'\,\mathbf{I}\right]\right)^-\left(\begin{bmatrix}\mathbf{Z}\\ \mathbf{I}\end{bmatrix}\mathbf{y}_2-\begin{bmatrix}\mathbf{Z}\\ \mathbf{I}\end{bmatrix}\mathbf{X}_2\boldsymbol{\beta}\right)\end{cases} \tag{3.2}$$

in the actual (2.1) and complete models (3.1) are equal;

c) The solutions of normal equations in the actual and complete ZM models are identical;

d) The sums of squares for error in the actual and complete ZM models are the same.

Result (d) is a generalization of Fisher's rule (1960).

4. Properties of predictors of missing observations in the general Gauss-Markov model

Another area of interest for the Professor was the research on the properties of predictors of missing observations. He studied the GM_d and ZM_d models with a diagonal covariance matrix $\mathbf{V}=\begin{bmatrix}\mathbf{V}_1,\mathbf{0}\\ \mathbf{0},\mathbf{V}_2\end{bmatrix}$, i.e. the models in which the available observations are uncorrelated with the missing observations.

The predictor $\hat{\mathbf{y}}_1=\mathbf{H}\mathbf{y}_2$ is called the UBLUP (uniformly best linear unbiased predictor) for $\mathbf{y}_1$ if for an arbitrary but fixed $\boldsymbol{\beta}$ is $E(\hat{\mathbf{y}}_1)=E(\mathbf{y}_1)=\mathbf{X}_1\boldsymbol{\beta}$ and one of the equivalent conditions is satisfied:

$$\underset{\tilde{y}_1=H_1y_2}{\forall}\left(E(\tilde{\mathbf{y}}_1)=E(\hat{\mathbf{y}}_1)\Rightarrow \operatorname{cov}(\tilde{\mathbf{y}}_1-\mathbf{y}_1)\geq \operatorname{cov}(\hat{\mathbf{y}}_1-\mathbf{y}_1)\right)$$

$$\underset{w}{\forall}\ \underset{\tilde{y}_1 = H_1 y_2}{\forall} \left(E(\tilde{\mathbf{y}}_1) = E(\mathbf{y}_1) \Rightarrow \mathrm{var}(\mathbf{w}'(\tilde{\mathbf{y}}_1 - \mathbf{y}_1)) \geq \mathrm{var}(\mathbf{w}'(\hat{\mathbf{y}}_1 - \mathbf{y}_1)) \right)$$

(Rao 1973, Silvey 1970).
In his work (Oktaba et al. 1985a) the Professor proved the following:

Theorem 4.1
In the GM_d model, the predictor $\hat{\mathbf{y}}_1 = \mathbf{H}\mathbf{y}_2$ is the UBLUP for $\mathbf{y}_1$ if and only if when for an arbitrary but fixed $\boldsymbol{\beta}$ is $E(\hat{\mathbf{y}}_1) = E(\mathbf{y}_1) = \mathbf{X}_1\boldsymbol{\beta}$ and one of the equivalent conditions is satisfied:

$$\underset{w}{\forall}\ \underset{\tilde{y} = H_1 y_2}{\forall} \left(E(\tilde{\mathbf{y}}_1) = E(\hat{\mathbf{y}}_1) \Rightarrow \mathrm{var}(\mathbf{w}'\tilde{\mathbf{y}}_1) \geq \mathrm{var}(\mathbf{w}'\hat{\mathbf{y}}_1) \right) \text{ or} \tag{4.1}$$

$$\underset{\tilde{y}_1 = H_1 y_2}{\forall} \left(E(\tilde{\mathbf{y}}_1) = E(\hat{\mathbf{y}}_1) \Rightarrow \mathbf{H}_1\mathbf{V}_2\mathbf{H}_1' - \mathbf{H}\mathbf{V}_2\mathbf{H}' \geq 0 \right). \tag{4.2}$$

Theorem 4.2
In the ZM_d model some predictors $\hat{\mathbf{y}}_1$ of missing observations $\mathbf{y}_1$ have the simple form:

$$\tilde{\mathbf{y}}_1 = \mathbf{X}_1(\mathbf{X}_2'\mathbf{V}_2^{-}\mathbf{X}_2)^{-}\mathbf{X}_2'\mathbf{V}_2^{-}\mathbf{y}_2 \tag{4.3}$$

They are invariant due to the choice of g-inverse $\mathbf{V}_2^{-}$ matrix.

The subclass of predictors as defined in (4.3) is denoted $\boldsymbol{\Psi}$. It is not difficult to observe that the predictors $\boldsymbol{\Psi}$ are of the form $\tilde{\mathbf{y}}_1 = \mathbf{X}_1\hat{\boldsymbol{\beta}}$, where $\hat{\boldsymbol{\beta}}$ is a solution of normal equations in the actual ZM_d model.

Theorem 4.3
If in the ZM_d model the following condition is satisfied:

$$R(\mathbf{X}_1') \subset R(\mathbf{X}_2'), \tag{4.4}$$

then the predictors of y_1 vector from $\boldsymbol{\Psi}$ class are UBLUP.

Theorem 4.4
In the ZMd model

$$\text{when } |\mathbf{V}| \neq 0 \text{ and X2 is of full column rank} \tag{4.5}$$

the sole predictor $\hat{\mathbf{y}}_1$ is UBLUP.

Conclusion 4.1
In the standard model $(\mathbf{y}, \mathbf{X\beta}, \sigma^2\mathbf{I})$ the predictor $\hat{\mathbf{y}}_1$ has the form:

$$\hat{\mathbf{y}}_1 = \mathbf{X}_1(\mathbf{X}_2'\mathbf{X}_2)^- \mathbf{X}_2'\mathbf{y}_2 \tag{4.6}$$

If in addition $R(\mathbf{X}_1') \subset R(\mathbf{X}_2')$ or $r(\mathbf{X}_2) = p$ then the $\hat{\mathbf{y}}_1$ predictor is UBLUP.

Theorem 4.5 (General form of UBLUP for y_1 in the GM_d model)
In the GM_d model $\hat{\mathbf{y}}_1 = \mathbf{H}\mathbf{y}_2$ is UBLUP for the vector $\mathbf{y}_1$ of missing observations if and only if:

$$R(\mathbf{X}_1') \subset R(\mathbf{X}_2') \quad \wedge \quad H = \mathbf{X}_1\mathbf{X}_2^- \quad \text{i.e.} \quad \hat{\mathbf{y}}_1 = \mathbf{X}_1\mathbf{X}_2^-\mathbf{y}_2 \tag{4.7}$$

$$\text{and } \underset{z}{\forall} \ \underset{w}{\forall} \, (z'X_2 = 0 \Rightarrow w'X_1X_2^-V_2z = 0)\,. \tag{4.8}$$

Observe that the condition (4.8) can equivalently be expressed as

$$R(\mathbf{V}_2\mathbf{H}') \subset R(\mathbf{X}_2)\,, \tag{4.9}$$

where the matrix **H** is defined in (4.7).

5. Invariant linearly sufficient transformations of the general Gauss-Markov model

Apart from the GM model $(\mathbf{y}, \mathbf{X\beta}, \sigma^2\mathbf{V})$ let us study the transformed PGM model of the form:

$$(\mathbf{Py}, \mathbf{PX\beta}, \sigma^2\mathbf{PVP}') \tag{5.1}$$

The question asked was as follows: Under which conditions on the transformation P in the GM and PGM models are there equal values of statistics related to the estimation and testing of the hypotheses in the linear model, namely: quadratic forms, sums of squares for error, the estimators of parametric functions $\boldsymbol{\lambda}'\boldsymbol{\beta}$ and of parameter σ^2 and test functions to verify the linear hypothesis? The answers to this question can be found in the Professor's works (Oktaba et al. 1984, Oktaba et al. 1988). The main results are presented below:

Theorem 5.1
If the linear transformation **P** meets one of the conditions:

$$R(\mathbf{T}) = R(\mathbf{TP}') \quad \wedge \quad R(\mathbf{X}) \subset R(\mathbf{P}'), \tag{5.2}$$

$$r(\mathbf{T}) = r(\mathbf{TP}') \quad \wedge \quad R(\mathbf{X}) \subset R(\mathbf{P}'), \tag{5.3}$$

$$R(\mathbf{T}) \subset R(\mathbf{P}'), \tag{5.4}$$

then the following values in models GM and PGM are equal:

a) quadratic forms:

$$(\mathbf{y} - \mathbf{X\beta})'\mathbf{T}^-(\mathbf{y} - \mathbf{X\beta}) \text{ and } (\mathbf{Py} - \mathbf{PX\beta})'(\mathbf{PTP}')^-(\mathbf{Py} - \mathbf{PX\beta}); \tag{5.5}$$

b) solutions of the normal equations;
c) conditions for estimability of parametric functions $\boldsymbol{\lambda}'\boldsymbol{\beta}$;
d) values of BLUEs for parametric functions $\boldsymbol{\lambda}'\boldsymbol{\beta}$;
e) estimators of parameter σ^2;
f) conditions for the consistency of the linear hypothesis $\mathbf{L\beta} = \boldsymbol{\varphi}_0$
g) test functions to test the linear hypothesis $\mathbf{L\beta} = \boldsymbol{\varphi}_0$

It should be noted that the conditions (5.2) and (5.3) are equivalent and (5.4) implies (5.2).

Transformations such as those described in Theorem 5.1 are called invariant linearly sufficient statistics and denoted as ILS (Oktaba et al. 1988). They retain the information needed for linear and quadratic estimation and testing in the linear model. ***They generalize the concept of linearly sufficient statistics***

(Baksalary and Kala 1981; Drygas 1983), which retain only the information needed for linear estimation

Specific forms of ILS appear in the problems related to the prediction of missing observations. These are for example: $\mathbf{P} = \begin{bmatrix} \mathbf{Z} \\ \mathbf{I} \end{bmatrix}$ transformation, which led from the actual to the complete model, permutations of observations or compositions of these transformations. Hence the next step in the research became tackling the issue of the reduction of missing observations freely placed in the multivariate models.

It should be noted that this presentation of the Professor's works on the Gauss-Markov and Zyskind-Martin models deals with the only part of his achievements in this field. Much of his works on projection operators and estimation and hypothesis testing in multivariate Gauss-Markov and Zyskind-Martin models has been published in national and international journals after his retirement (Oktaba 1998, 2002). Some of the Professor's results used in this paper have been and still are quoted in world literature on the subject (e.g. in the international *Encyclopedia of Statistical Sciences*: Kotz et al. 1988; Alexander, Chandrasekar 2005; Stepniak 2005). They have also been used by the authors of this paper in their works, e.g. on multivariate models and the application of the transformation of nonlinear models (Kornacki 2007, Wawrzosek 2009). Numerous text books written by the Professor are still quoted in various publications.

References

Aitken, A. C. (1934). On least squares and linear combination of observations. *Proc. Roy. Soc. Edinb*. A, 55, 42-7.

Alexander T.L., Chandrasekar B. (2005). Simultaneous equivariant estimation of the parameters of matrix scale and matrix location-scale models. *Statistical Papers*, 46, 4, 483-507.

Baksalary J.K, Kala R. (1981). Linear transformations preserving best linear unbiased estimators in a general Gauss-Markoff model. *Ann. Statist*. **9**, 913-916.

Drygas H. (1983). Sufficiency and completeness in the general Gauss-Markoff model. *Sankhya* Ser A. 45, 88-99.

Fisher R.A. (1960). *The design of experiments*. Edinburgh.

Kornacki A. (2007). Different kinds of sufficiency in the general Gauss-Markov model. *Mathematica Slovaca*, 57, 4, 389-392.

Kotz S., Johnson N.L., Read C.R. (1988). *Encyclopedia of Statistical Sciences*, vol. 9, 683.

Oktaba W. (1982a). Testy hipotezy liniowej w stałych modelach liniowych z dodatnio określoną macierzą kowariancji. *XII Colloquium z Agrobiometrii,* PAN, Warszawa, 34-79.

Oktaba W. (1982b). Testy hipotezy liniowej w stałych modelach liniowych z nieujemnie określoną macierzą kowariancji. *XII Colloquium z Agrobiometrii*, PAN, Warszawa, 80-122.

Oktaba W., Jagiełło G. (1982). Niestandardowe modele liniowe z brakującymi obserwacjami. *XII Colloquium z Agrobiometrii*, PAN, Warszawa, 142-161.

Oktaba W., Kornacki A., Wawrzosek J. (1983). Estymacja brakujących wyników w ogólnym modelu Gaussa-Markowa. *XIII Colloquium z Agrobiometrii*, PAN, Warszawa, 118-218.

Oktaba, W. (1984). Tests of hypotheses for the general Gauss-Markov model. *Biom. J.* 26, 415-424.

Oktaba W., Kornacki A., Wawrzosek J. (1984). Niezmiennicze transformacje liniowe ogólnego modelu Gaussa-Markowa. Estymacja i testowanie. *XIV Colloquium z Agrobiometrii*, PAN, Warszawa, 124-134.

Oktaba W., Kornacki A., Wawrzosek J. (1985a). Jednostajnie najlepsze liniowe i nieobciążone estymatory brakujących obserwacji w pewnym modelu liniowym Gaussa-Markowa. *XV Colloquium z Agrobiometrii*, PAN, Warszawa, 124-134.

Oktaba W., Kornacki A., Wawrzosek J. (1985b). Estimation and verification of hypotheses in some Zyskind-Martin models with missing values. *Biom. J.* 7, 733-740.

Oktaba W., Kornacki A., Wawrzosek J. (1986). Estimation of missing values in the general Gauss-Markoff model. *Statistics* 17, 167-177.

Oktaba W., Kornacki A., Wawrzosek J. (1988). Invariant linearly sufficient transformations of the general Gauss-Markoff model. Estimation and testing. *Scand. J. Statist* 15, 117-124.

Oktaba W. (1998). Characterization of the multivariate Gauss-Markoff model with singular covariance matrix and missing values. *Applications of Mathematics*, Vol. 43, No. 2, 119—131.

Oktaba W. (2002). *Historia teorii eksperymentu.* Lubelskie Towarzystwo Naukowe.

Rao C.R. (1972). Some recent results in linear estimation. *Sankhyā*, Series B, Vol.34, 369-378.

Rao C.R. (1973). *Linear Statistical Inference and its Applications*. Wiley Series in Probability and Statistics, New York.

Silvey S.D. (1970). *Statistical inference*. London, Chapman and Hall.

Stępniak Cz. (2005). Matrix equations with restraints and their statistical applications. *Linear Algebra and its Applications* 410, 279-287 Tenth Special Issue (Part 2) on Linear Algebra and Statistics.

Wawrzosek J. (2009). *Modelowanie statystyczne zużycia paliwa i emisji spalin w silnikach ciągnikowych zasilanych biopaliwem.* Praca monograficzna, Wydawnictwo Uniwersytetu Przyrodniczego w Lublinie.

Yates F. (1933). The analysis of replicated experiments when the field results are incomplete. *Emp. Jour. Exp. Agr.* 1, 129-142.

Zyskind G., Martin F. B. (1969). On best linear estimation and a general Gauss-Markov theorem in linear models with arbitrary nonnegative covariance structure. *SIAM J. Appl. Math.* 1190-1202.