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THE ENGESSER-SHANLEY MODIFIED THEORY OF STABILITY OF THIN-WALLED CYLINDRICAL RODS WITH EXAMPLE OF USE FOR STEEL ST 35

Krzysztof Murawski

Abstract: The Engesser-Shanley modified theory of thin-walled axially crushed cylindrical rods is presented in the paper. The problems of critical loads in elasto-plastic states have been considered. The plasticity ratio of critical transverse section has been defined by value α-angle, which to slenderness ratio of rod is related. The received theoretical results to the experience research effects of rods made of steel St 35 have been compared.

Key words: stability, critical load, thin-walled, Engesser, Kármán, Shanley

STABILITY OF RODS ACCORDING TO ENGESSER, KÁRMÁN AND SHANLEY HYPOTHESISES

The basic theory of slender rod lose stability in elastic state was worked out through Euler [1744, 1759]. He introduced the concept of critical load and gave the formula for critical force of axially crushed rod. The loss stability theory of crushed rod in elasto-plastic state, basing on concept of tangent module, worked out Engesser [1889, 1895], Kármán [1908, 1910] and Shanley [1947].

According to Engesser-Kármán hypothesises, in critical cross-section of axially crushed rod, being found in elasto-plastic state, there are two zones: squeezed – deformed plastically on concave side of neutral layer of rod and tensioned – deformed elastically on bossed side (Fig. 1A). From equilibrium of forces and of moments in relation to neutral layer of rod, before losing of stability, drew out the conclusion, that areas of triangles of tensioning and squeezing stresses have to be equal. Besides elastic zone characterised with Young's module E and plastic zone with so-called tangent module E_t, determined as Young's module from graph stress σ – strain ε, received during standard tension test, but from non-linear range.

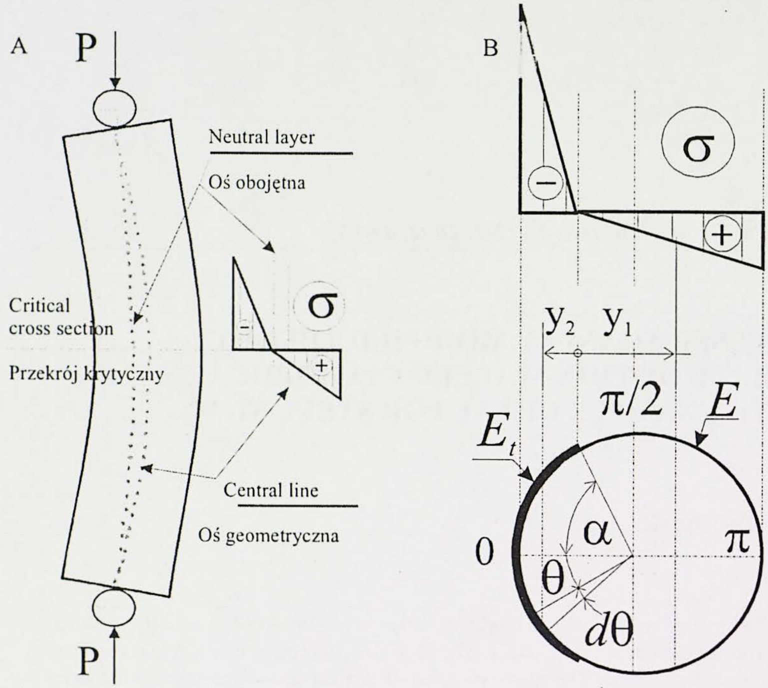

Fig. 1. Axially crushed rod (A) and its critical cross-section (B) while losing of stability according to Engesser-Kármán hypothesises
Rys. 1. Osiowo ściskany pręt (A) i jego przekrój krytyczny (B) podczas utraty nośności według hipotezy Engessera- Kármána

Referring this hypothesises to case of cylindrical rod (Fig. 1B), where in critical cross-section, y_1 and y_2 – are distances adequately from neutral layer to segment dA of elastic and plastic zone, in moment of losing stability, can be expressed:

$$dA = t \cdot \overline{R} \cdot d\theta, \quad y_1 = \overline{R} \cdot (\cos\alpha - \cos\theta), \quad y_2 = \overline{R} \cdot (\cos\theta - \cos\alpha) \tag{1}$$

where: $\overline{R}$ – the average radius of tube,
t – the wall thickness,
2α – angle describing plastic part critical cross-section,
A – area of critical cross-section.

The equilibrium of forces from stresses, in relation to neutral layer, described with formulas:

$$\int_{A_2} \sigma_t \, dA_2 = \int_{A_1} \sigma_1 \, dA_1 \;, \quad \sigma_t = \frac{y_2}{\rho} \cdot E_t \,, \quad \sigma_1 = \frac{y_1}{\rho} \cdot E \tag{2}$$

where: A_1, A_2 – area adequately of tensioned and plastic part of critical transverse cross section,
ρ – curvature radius of neutral layer.

From Equations (1) and (2):

$$\int_{\alpha}^{\pi} \frac{E_t}{\rho} \cdot \bar{R} \cdot (\cos\alpha - \cos\theta) \cdot t \cdot \bar{R} d\theta = \int_{0}^{\alpha} \frac{E}{\rho} \cdot \bar{R} \cdot (\cos\theta - \cos\alpha) \cdot t \cdot \bar{R} d\theta \quad (3)$$

and after integrating:

$$\frac{E_t}{E} = \frac{\sin\alpha + (\pi - \alpha) \cdot \cos\alpha}{\sin\alpha - \alpha \cdot \cos\alpha} = 1 + \frac{\pi}{\operatorname{tg}\alpha - \alpha} \quad (4)$$

Function of Equation (4) is represented on Figure 2, from which is apparent, that it has physical sense only for $\alpha \in (90°, 180°)$, because should be: $1 \geq E_t / E > 0$. At $\alpha = 90°$ is $E_t = E$, i.e. that whole critical cross-section is in elastic state, what answers to limiting slenderness ratio – λ_{gr}. Would result from here, that for slenderness ratio λ little smaller than λ_{gr}, follows suddenly plasticization of half critical cross-section ($\alpha = 90°$), and then for more and more smaller λ plastic zone grows until gets angle 180° attaining full plastic state. On the base of research works observations has been ascertained, that it is not consistent with reality, because the plastic area, described with angle α, there is also for $\alpha < 90°$.

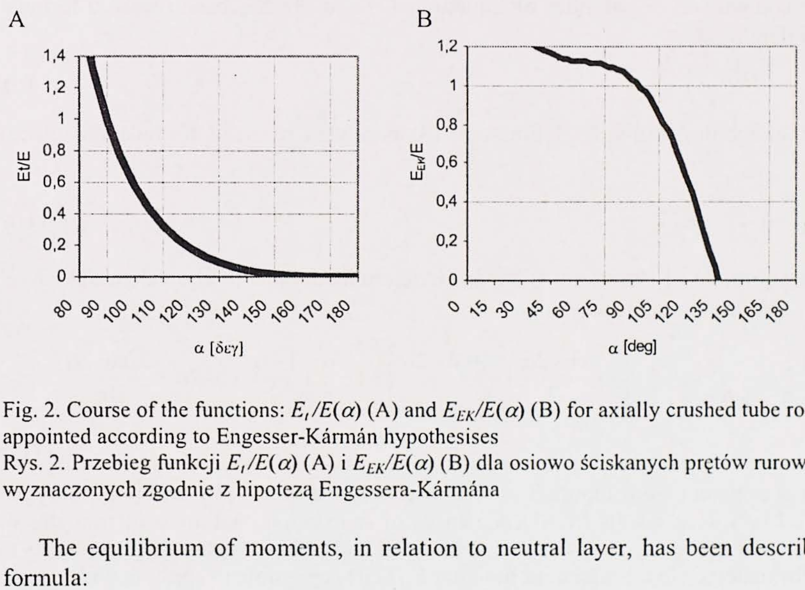

Fig. 2. Course of the functions: $E_t/E(\alpha)$ (A) and $E_{EK}/E(\alpha)$ (B) for axially crushed tube rods appointed according to Engesser-Kármán hypothesises

Rys. 2. Przebieg funkcji $E_t/E(\alpha)$ (A) i $E_{EK}/E(\alpha)$ (B) dla osiowo ściskanych prętów rurowych wyznaczonych zgodnie z hipotezą Engessera-Kármána

The equilibrium of moments, in relation to neutral layer, has been described with formula:

$$\int_{A_2} \sigma_t \cdot y_2 \, dA_2 + \int_{A_1} \sigma_1 \cdot y_1 \, dA_1 = -P \cdot y \quad (5)$$

where y – distance from outside load line to neutral layer.

After integrating:

$$\frac{E_t}{\rho}\cdot \overline{R}^3 \cdot t\left[\frac{3}{2}\alpha\cdot\sin 4\alpha\right]+\frac{E}{\rho}\cdot\overline{R}^3\cdot t\left[(\pi-\alpha)\cdot\left(1+\frac{\cos 2\alpha}{2}\right)+\frac{3\cdot\sin 2\alpha}{4}\right]=-\frac{1}{2}P\cdot y \qquad (6)$$

and taking into consideration in critical cross-section J_1, J_2 – adequately axial moment of inertia of tensioned and plastic part, has been received:

$$\frac{E_t J_2 + EJ_1}{\rho} = -P\cdot y \qquad (7)$$

Analysing differential equation of neutral layer deflection line relatively to outside load line (on the assumption of little deformations $\ddot{y}\approx\frac{1}{\rho}$ and of equivalent module E_{EK} existence) has been received neutral layer deflection line equation:

$$E_{EK}\ J\cdot\ddot{y}+P\cdot y=0 \qquad \Rightarrow \qquad \frac{E_{EK}J}{\rho}=-P\cdot y \qquad (8)$$

After comparison of left sides of Equations (7) and (8), has been received formula for equivalent module:

$$E_{EK}=\frac{E_t J_2 + EJ_1}{J} \qquad (9)$$

After integrating differential Equation (8) has been received formula on critical stress:

$$\sigma_{EK}=\left(\frac{\pi}{\lambda}\right)^2\cdot E_{EK} \qquad (10)$$

Taking formula of Equation (4) has been determined value of equivalent module in relation to Young's module as:

$$\frac{E_{EK}}{E}=\left(\frac{1}{\pi}+\frac{1}{\operatorname{tg}\alpha-\alpha}\right)\cdot\left[\alpha\left(1+\frac{\cos 2\alpha}{2}\right)-\frac{3\cdot\sin 2\alpha}{4}\right]+1-\frac{\alpha}{\pi}+\frac{1-\alpha}{2}\cdot\cos 2\alpha+\frac{3\cdot\sin 2\alpha}{4\pi} \qquad (11)$$

How is apparent from Figure 3 course of this function has physical point only for $\alpha\in(90°, 138°)$ what results from acceptances of foundation, that areas of triangles of tensioning and squeezing stresses have to be equal and from limitations of hypothesis to little deformations. How is apparent function $E_t/E(\alpha)$ is possible to apply in wider range of α than $E_{EK}/E(\alpha)$. Shanley took advantage of this putting in Equation (10) instead E_{EK} directly E_t, what also was burdened with an error, and appointed course of function $E_t(\lambda)$ on the ground of this formula at acquaintances from experiment the function critical stress σ_{kr} – slenderness ratio λ is rather theoretical. Especially wakes reservations determining of function $E_t(\lambda)$ for row of rods on the ground of non-linear range of function $\sigma(\varepsilon)$ course, received during extension tests of one normative samples [Březina 1966; Wolmir 1967].

STABILITY OF RODS IN OWN INVESTIGATION

In connection with above reservations has been conducted own analysis of stability of thin-walled rods. First results of this were presented early [Murawski 1992]. Have been made research works on tube which was crushed through ball and socket joints with constants increase. Strains from strain gauges, have been presented on Figure 3.

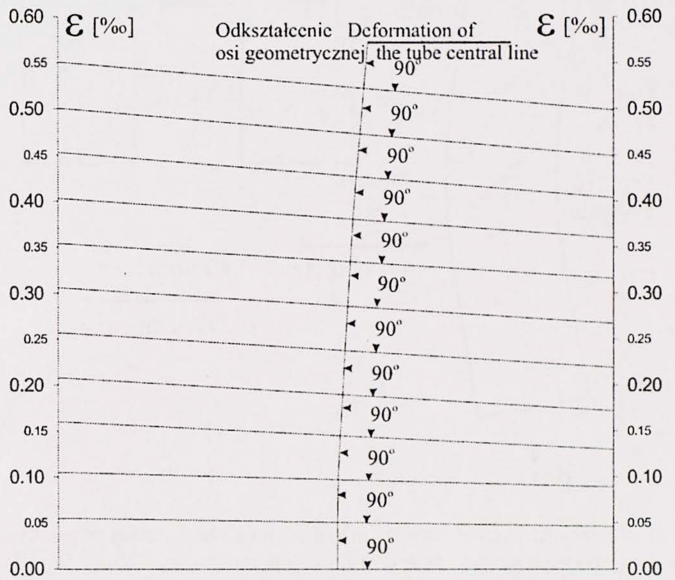

Fig. 3. Strains from strain gauges, placed in central cross-section of tube rod on two opposite generating lines and appointed on this base a turnover of central segment of tube geometrical central line (the tube rod weighted through ball and socket joints with constants increase of charge)

Rys. 3. Odkształcenia względne otrzymane z tensometrów umieszczonych w środkowym przekroju na dwu przeciwległych tworzących i wyznaczony na tej podstawie obrót osi geometrycznej tulei cylindrycznej (obciążanej przez przeguby stałym przyrostem obciążenia)

Results from this figure, that already little charges, next to 10% P_{kr}, cause distinct turnover of centre line tube segment and testify to its deforming. It has been accepted so, that losing of stability follows already at minimum charges, and for losing of carrying capacity is responsible position of resultant (from superposition of compressions and of bending) neutral layer of tube in relation to critical outline of transverse section. On base of these observations it has been accepted, that in elastic state losing of carrying capacity follows after exit resultant neutral layer from critical transverse section, however in elasto-plastic state, after entry resultant neutral layer in plastic zone. It has been accepted from here, that state of stresses in critical transverse section after loss of stability and before loss of carrying capacity, is as the result of clean compressions and bending superposition (Fig. 4).

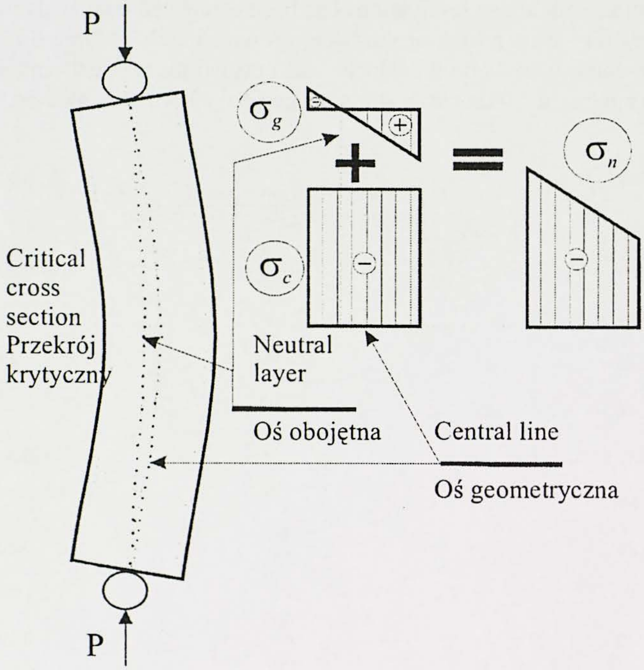

Fig. 4. State of stresses after loss of stability in critical transverse section of axially crushed tube according to own investigation. σ_c – stress from compressions, σ_g – stress from bending, σ_n – normal stress (superposition of compressions and of bending); there are not shear stress in critical transverse section

Rys. 4. Stan naprężeń po utracie stateczności w krytycznym przekroju poprzecznym rury ściskanej osiowo zgodnie z własną hipotezą: σ_c – naprężenie ściskające, σ_g – naprężenie gnące, σ_n – naprężenie normalne (superpozycja naprężeń ściskających i gnących); w krytycznym przekroju poprzecznym nie występują naprężenia gnące

As one can notice from drawing, in critical transverse section the most strenuous, because of normal stresses, is extreme internal filament after concave rod side. Because of clean compressions and bending superposition resultant neutral layer shifts, together with growth of load, from central line toward concave, more weighted side of tube. Together with growth of load, grow up stresses from bending, but also from compressions, and the position of resultant neutral layer is as the effect of these stresses participation specific game, what depends from the form of rod (thickset – slender). State of stresses in critical transverse section of rod, before loss of carrying capacity, is characterised with flow plastic part of section, in concave squeezed side, after stresses rich yield stress. This results with additional decreasing of rod central line radius. On Figure 5A the position of neutral layer in critical transverse section before loss of carrying capacity has marked with angle β and position of plastic zone border – with angle α. As it was stated before it, has been accepted, that loss of carrying capacity of rod follows when resultant neutral layer, at force P_{kr}, enters in plastic zone, that is $\alpha = \beta$ (Fig. 5B) and flowing material does not place already resistance against bending (the lack of resisting moment).

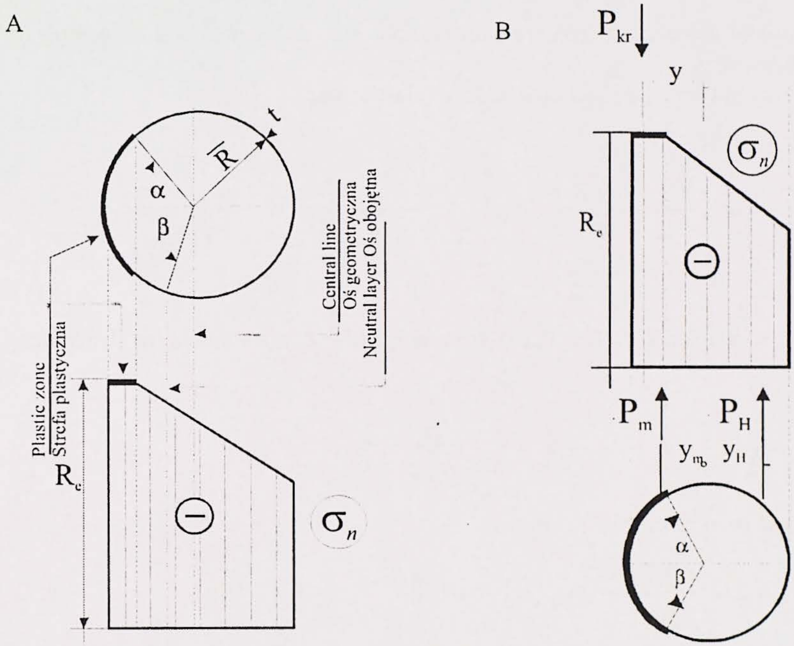

Fig. 5. State of stresses before (A) and in moment (B) of loss of carrying capacity in critical transverse section of axially crushed tube

Rys. 5. Stan naprężeń przed (A) i w momencie (B) utraty nośności w krytycznym przekroju poprzecznym rury ściskanej osiowo

For λ_l has been accepted, that in moment of loss of carrying capacity full critical transverse section is in plastic state ($\alpha = \pi$). In case of λ_{gr} whole critical transverse section, in the moment of loss of carrying capacity, is in plastic state ($\alpha = 0$). The position of resultant neutral layer, in the moment of loss of carrying capacity, changes dependably on slenderness ratio. For the rod with slenderness ratio λ_l the resultant neutral layer covers the central line ($\alpha = \pi$, $\beta = \pi/2$) and with λ_{gr} covers the edge of critical transverse section outline ($\alpha = 0$, $\beta = 0$). For slenderness ratio between λ_l and λ_{gr} the resultant neutral layer lies between the central line and the edge of critical transverse section outline on concave side of rod ($0 < \alpha < \pi$; $\pi/2 > \beta > 0$). After cutting the rod in the critical cross-section and after investigation of state of stresses equilibrium of forces has been qualified. The force P_{kr} is equalised by elastic force P_H and plastic force P_m (Fig. 5B). Therefore [Murawski 1992]:

$$\Sigma P = P_{kr} - P_m - P_H = 0 \Rightarrow P_{kr} = P_m + P_H \tag{12}$$

$$\Sigma M = P_{kr} \cdot y - P_m \cdot y_m + P_H \cdot y_H = 0 \tag{13}$$

where:

$$P_H = \sigma_H \cdot A_H = \sigma_H \cdot (\pi - \alpha) \cdot \overline{R} \cdot t, \quad P_m = \sigma_{kr}(\lambda_1) \cdot A_m = \sigma_{kr}(\lambda_1) \cdot \alpha \cdot \overline{R} \cdot t \tag{14}$$

σ_H – limit of elasticity at compression (variable for different λ and answering to them α),

$A_{H(m)}$ – area of transverse section in elastic (plastic) state.

From here:

$$\sigma_{kr} = \frac{P_{kr}}{A} = \sigma_H \frac{2 \cdot (\pi - \alpha) \cdot \overline{R \cdot t}}{2 \cdot \pi \cdot \overline{R \cdot t}} + \sigma_{kr}(\lambda_1) \cdot \frac{2 \cdot \alpha \cdot \overline{R} \cdot t}{2 \cdot \pi \cdot \overline{R \cdot t}} = \sigma_H \left(1 - \frac{\alpha}{\pi}\right) + \sigma_{kr}(\lambda_1) \cdot \frac{\alpha}{\pi} \tag{15}$$

If for λ_{gr} is attained P_H, then $P_{kr}(\lambda)$ from range (λ_1, λ_{gr}) is attained for slenderness ratio:

$$\lambda = \lambda_{gr} - \frac{\alpha}{\pi}\left(\lambda_{gr} - \lambda_1\right) \quad \Rightarrow \quad \frac{\alpha}{\pi} = \frac{\lambda_{gr} - \lambda}{\lambda_{gr} - \lambda_1} \tag{16}$$

From here has been received:

$$\sigma_{kr}(\lambda) = \sigma_H(\lambda) + \frac{\lambda_{gr} - \lambda}{\lambda_{gr} - \lambda_1} \cdot \left[\sigma_{kr}(\lambda_1) - \sigma_H(\lambda)\right] \tag{17}$$

The stress $\sigma_{kr}(\lambda_1)$ has beenacknowledged as characteristic parameter for given material and marked Re^*. From researches has resulted, that function $\sigma_H(\lambda)$ is of first degree, so received function $\sigma_{kr}(\lambda)$ is of second degree. The stress $\sigma_H(\lambda_1)$ has been acknowledged too as characteristic parameter and marked R^*_H. The linear function $\sigma_H(\lambda)$, neglecting value λ_1, has been described with formulas:

$$\sigma_H(\lambda) = R_H^{Eu} + \left(1 - \frac{\lambda}{\lambda_{gr}}\right) \cdot \left[R_H^* - R_H^{Eu}\right], \qquad R_H^{Eu} = \sigma_H(\lambda_{gr}) = \sigma_{kr}(\lambda_{gr}) \tag{18}$$

where R_H^{Eu} – limit of elasticity practical in Euler's formula to determination λ_{gr}.

Then:

$$\sigma_{\mathrm{kr}}(\lambda)=\left(1-\frac{\lambda}{\lambda_{\mathrm{gr}}}\right)\cdot\left(R_e^{*}+R_H^{*}\frac{\lambda}{\lambda_{\mathrm{gr}}}\right)+R_H^{Eu}\left(\frac{\lambda}{\lambda_{\mathrm{gr}}}\right)^2 \tag{19}$$

After insertion the Euler's formula:

$$\sigma_H(\lambda)=\left(\frac{\pi}{\lambda_{\mathrm{gr}}}\right)^2\cdot E+\left(1-\frac{\lambda}{\lambda_{\mathrm{gr}}}\right).\left[R_H^{*}-\left(\frac{\pi}{\lambda_{\mathrm{gr}}}\right)^2 E\right]$$

$$\sigma_{\mathrm{kr}}(\lambda)=\left(1-\frac{\lambda}{\lambda_{\mathrm{gr}}}\right)\cdot\left(R_e^{*}+R_H^{*}\cdot\frac{\lambda}{\lambda_{\mathrm{gr}}}\right)+E\cdot\left(\frac{\pi\cdot\lambda}{\lambda_{\mathrm{gr}}^2}\right)^2 \tag{20}$$

or:

$$\sigma_H(\lambda)=R_H^{Eu}+\left(1-\frac{\lambda}{\pi}\sqrt{\frac{R_H^{Eu}}{E}}\right)\cdot\left(R_H^{*}-R_H^{Eu}\right),$$

$$\sigma_{\mathrm{kr}}(\lambda)=\left(1-\frac{\lambda}{\pi}\cdot\sqrt{\frac{R_H^{Eu}}{E}}\right)\cdot\left(R_e^{*}+R_H^{*}\cdot\frac{\lambda}{\pi}\cdot\sqrt{\frac{R_H^{Eu}}{E}}\right)+\frac{1}{E}\cdot\left(\frac{\lambda}{\pi}\cdot R_H^{Eu}\right)^2$$

These formulas are suitable to use on condition acquaintance of parameters: R_e^{*}, R_H^{*}, E, R_H^{Eu} or λ_{gr}. Two first you should determine from the compression test of thickset rod. The comparisons of graphs of theoretical function and of approximated functions from own researches are presented on Figure 6 (R_e^{*} = 624.82 MPa, R_H^{*} = = 422.86 MPa, $E=211\ 000$ Mpa, $R_H^{Eu}=219.5$ MPa). The differences of values of those stresses functions vary with slenderness ratio but does not exceeds 11.4%.

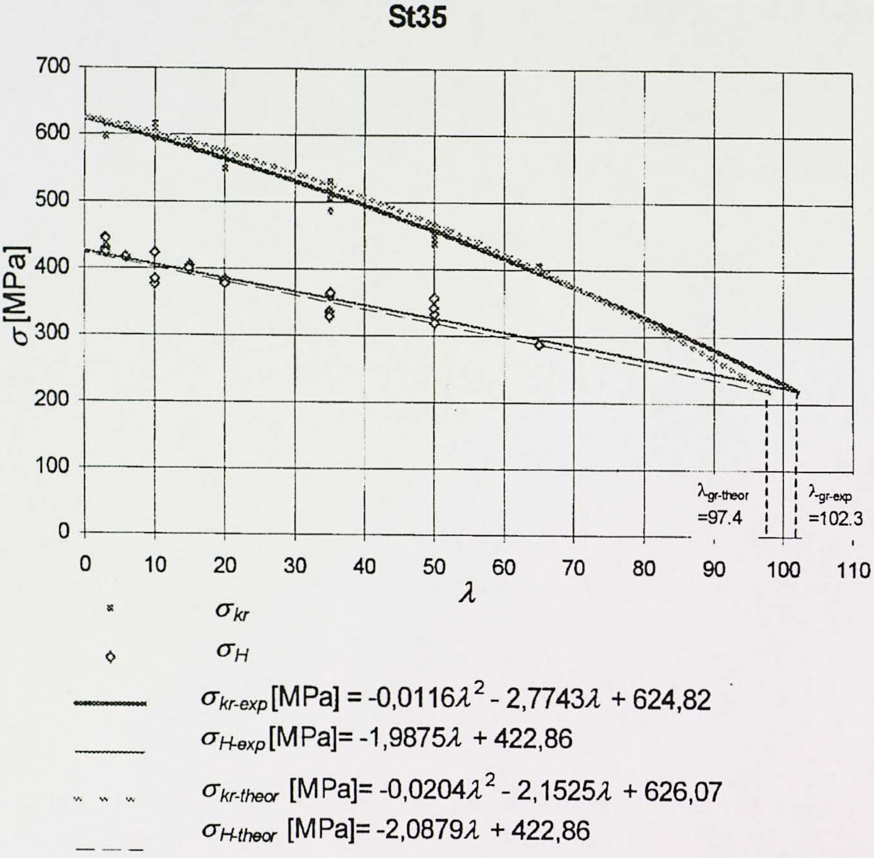

Fig. 6. Comparison of graphs of the theoretical function and of approximated functions from researches

Rys. 6. Porównanie wykresów funkcji teoretycznych z aproksymowanymi funkcjami z badań

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ZMODYFIKOWANA TEORIA STATECZNOŚCI ENGESSERA-SHANLAY'A DLA CYLINDRYCZNYCH CIENKOŚCIENNYCH PRĘTÓW Z PRZYKŁADEM ZASTOSOWANIA DLA STALI ST35

Streszczenie: W pracy przedstawiono zmodyfikowaną teorię Engessera-Shanlay'a w odniesieniu do osiowo ściskanych cienkościennych prętów cylindrycznych. Rozpatrywano problem obciążeń krytycznych w stanach sprężysto-plastycznych. Wpływ uplastycznienia krytycznego przekroju poprzecznego określono kątem α, który odnoszono do smukłości pręta. Otrzymane teoretyczne wyniki porównano z wynikami badań eksperymentalnych na próbkach ze stali St35.

Slowa kluczowe: stateczność, obciążenie kryrtczne, cienkościenny, Engesser, Kármán, Shanley

K. Murawski, Department of Machine Design Fundamentals, Agricultural University of Poznan, Wojska Polskiego 38/42, 60-627 Poznan