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On determining the nonnegative relaxation spectrum of viscoelastic materials using complementary error functions

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Summary. The relaxation spectrum is a characteristic quantity describing the viscoelastic properties of many materials. Given the spectrum, it is very easy to convert one material function into another one. The paper deals with the problems of recovery of the spectrum of relaxation frequencies of linear viscoelastic materials from discrete-time noise corrupted measurements of relaxation modulus obtained in stress relaxation test. A new concept of the approximation of continuous relaxation spectrum by the finite series of nonnegative power basis functions is presented. It is proved that the respective model of the relaxation modulus is given by finite series of functions based on complementary error function. The optimal scheme of the least-squares identification of nonnegative definite model of relaxation spectrum is proposed. The validity of the method is demonstrated using simulated data of Gaussian spectrum. Applying the proposed scheme, the relaxation spectrum of a confined cylindrical specimen of the sugar beet root is also determined.
Key words: viscoelasticity, relaxation spectrum, identification, error functions.

INTRODUCTION

The last three decades have witnessed an increasing amount of interest paid to the study on mechanical properties of viscoelastic materials treating it as a separated subject area [11,13]. Viscoelastic models are used before all to modeling of different polymeric liquids and solids [3,11,13], concrete [13], soils and rocks [9], different composite materials [13], metals and their alloys [13,21], foods [13,16] and many biological materials [13,14], in particular fruits and vegetables [7,10,13,14,18]. The need for detailed knowledge of mechanical material functions has been growing with the increased use of accurate engineering methods for rigorous predictions of the materials behavior, such as the finite element method, the boundary element method and the finite difference method [1].

The mechanical properties of linear viscoelastic materials are characterized by relaxation or retardation spectra [3,13]. The spectra are vital not only for constitutive models but also for the insight into the properties of a viscoelastic material, since from the relaxation or retardation spectrum any other linear material functions can be calculated without difficulty. Hence the identification of relaxation spectrum is one of the actual directions of the mathematical modelling of many viscoelastic materials. Since the problem of relaxation spectrum identification is one of the classical ill-posed inverse problems [13,18], an appropriate special identification methods must be find to overcome the mathematical difficulties.

A number of different methods have been proposed during the last two decades for the relaxation spectrum computation using the data from a small-amplitude oscillatory shear experiment. For references and an overview, see Dealy and Larson [3]. However, a classical manner of studying viscoelasticity for many materials is by two-phase stress relaxation test [13,17]. There are a few papers: [5,23] and [18-20] as well as the previous papers by the present author cited therein, that deal with the spectrum determination from stress relaxation data. However, in all the known methods of the relaxation spectrum identification from time-measured data the restriction that the spectrum is nonnegative, which must be given to satisfy the physical meaning is, unfortunately, neglected. The identification scheme presented in this paper overcomes this limitation.

In this paper the spectrum is recovered from discrete-time noise corrupted measurements of relaxation modulus obtained in stress relaxation test. The approach proposed is based on approximation of the spectrum by finite linear combination of the basis exponential functions. A quadratic identification index, which refers to the measured relaxation modulus, is adopted and the nonnegativity constraints on the model parameters are introduced. The stated linear least-squares approximation task with inequality constraints is solved by using the dual approach. The result-

ing identification scheme is presented and verified for numerical data as well as for selected biological material.

RELAXATION SPECTRUM

In the rheological literature it is commonly assumed that the relaxation modulus $G(t)$ of linear viscoelastic material has the following integral representation [13,20]:

$$G(t)=\int_0^\infty H(\nu)\,e^{-t\nu}d\nu\ , \tag{1}$$

where: the nonnegative relaxation spectrum $H(\nu)$ characterizes the distribution of relaxation frequencies $\nu\geq 0$. Throughout we shall be concerned with the case when $H(\nu)$ is completely unknown. The relaxation modulus $G(t)$ can be, however, measured for any time $t\geq 0$.

MODELS

Assume that $H(\nu)\in L_2(0,\infty)$, where $L_2(0,\infty)$ is the space of square-integrating functions in the interval $(0,\infty)$. The set of the linearly independent functions:

$$\left\{e^{-\alpha\nu^2},\nu e^{-\alpha\nu^2},\nu^2 e^{-\alpha\nu^2},\dots\right\},\quad \alpha>0 \tag{2}$$

forms a basis of the space $L_2(0,\infty)$ [2]. Thus the relaxation spectrum can be expressed as $H(\nu)=\sum_{k=0}^{\infty} g_k\,h_k(\nu,\alpha)$, where: g_k are constant model parameters and the basis functions $h_k(\nu,\alpha)=\nu^k e^{-\alpha\nu^2}$, $k=0,1,\dots$. Since for real materials the relaxation spectrum tends to zero as the relaxation frequency approaches zero from above, i.e., the right-sided limit $lim_{\nu\to 0^+}H(\nu)=0$, the first basis function can be neglected. For practical reasons, it is convenient to replace the infinite summation in the above formula with a finite one of K terms of the form: $\sum_{k=1}^{K} g_k\,h_k(\nu,\alpha)$. The norms of the basic functions $h_k(\nu,\alpha)$ are given by:

$$\|h_k\|_2^2=\int_0^\infty \nu^{2k}e^{-2\alpha\nu^2}\,d\nu=\frac{\prod_{j=1}^{k}(2j-1)}{2^{2k+1}\alpha^k}\sqrt{\frac{\pi}{2\alpha}}\ . \tag{3}$$

Here $\|\cdot\|_2$ denotes the square norm in the space $L_2(0,\infty)$. It is easy to check that for any fixed α the sequence of norms (3) increases or decreases very quickly depending on the parameter α. Thus, in order to guarantee the numerical stability of the resulting identification algorithm the basis functions (2) are replaced by following normalized functions:

$$\tilde{h}_k(\nu,\alpha)=\gamma_k(\alpha)\nu^k e^{-\alpha\nu^2}\quad for\quad k=1,2,\dots, \tag{4}$$

where, by virtue of (3), the weights γ_k are defined by:

$$\gamma_k(\alpha)=2^k\sqrt{2\alpha^k}\,\sqrt[4]{\frac{2\alpha}{\pi}}\Big/\sqrt{\prod_{j=1}^{k}(2j-1)}\quad for\quad k=1,2,\dots. \tag{5}$$

Thus the finite sum:

$$H_K(\nu)=\sum_{k=1}^{K} g_k\,\tilde{h}_k(\nu,\alpha) \tag{6}$$

will be used to approximate the relaxation spectrum $H(\nu)$. The lower index of $H_K(\nu)$ is the number of model summands. According to equation (1) the respective model of the relaxation modulus $G(t)$ is described by:

$$G_K(t)=\int_0^\infty H_K(\nu)e^{-t\nu}\,d\nu=\sum_{k=1}^{K} g_k\,\phi_k(t,\alpha), \tag{7}$$

where:

$$\phi_k(t,\alpha)=\int_0^\infty \tilde{h}_k(\nu,\alpha)e^{-t\nu}\,d\nu. \tag{8}$$

The form of the basis functions $\phi_k(t,\alpha)$ (8) is given by the following theorem; the proof is omitted due to space limitations.

Theorem 1. *Let $k\geq 0$, $\alpha>0$ and $t\geq 0$. Then the basis functions $\phi_k(t,\alpha)$ are given by recursive formula:*

$$\phi_{k+1}(t,\alpha)=\frac{2}{\sqrt{2k+1}}\frac{k}{\sqrt{2k-1}}\left[\phi_{k-1}(t,\alpha)-\frac{\sqrt{2k-1}}{2k\sqrt{\alpha}}t\phi_k(t,\alpha)\right],\quad k\geq 1, \tag{9}$$

starting with:

$$\phi_0(t,\alpha)=\sqrt[4]{\frac{\pi}{2\alpha}}\,e^{\frac{t^2}{4\alpha}}\,erfc\left(\frac{t}{2\sqrt{\alpha}}\right), \tag{10}$$

$$\phi_1(t,\alpha)=2\sqrt[4]{\frac{1}{2\alpha\pi}}-\frac{t}{\sqrt{\alpha}}\phi_0(t,\alpha), \tag{11}$$

where: $erfc(t)=\frac{2}{\sqrt{\pi}}\int_t^\infty e^{-x^2}dx$ is complementary error function [4].

Thus, the problem of the continuous relaxation spectrum $H(\nu)$ approximation by finite series $H_K(\nu)$ of the form (6) is reduced to problem of the relaxation modulus $G(t)$ approximation by finite linear combination (7) of the functions $\phi_k(t,\alpha)$ (9)-(11) based on complementary error function $erfc\left[t/\left(2\sqrt{\alpha}\right)\right]$. Error functions are used in signal processing and mathematical modelling of many processes, e.g., [4, 22]. A few first basis functions $\tilde{h}_k(\nu,\alpha)$ (4) are shown in Figure 1 for two different values of the parameter α; the corresponding functions $\phi_k(t,\alpha)$ (9)-(11) are plotted in Figure 2. From the Figure 2, it is evident that the

basis functions $\phi_k(t,\alpha)$ are congruent to the real relaxation modulus obtained in experiment. The positive parameter α is a time-scaling factor.

It is well-known [13,18,19] that for many materials $lim_{t\to\infty} G(t) = G_\infty > 0$, where G_∞ is the long-term modulus (see example 2 below). Thus, instead of the classical model (1), it is convenient to consider the following augmented model:

$$\overline{G}_K(t) = \int_0^\infty H_K(\nu) e^{-t\nu} d\nu + G_\infty = G_K(t) + G_\infty. \quad (12)$$

Then, the relaxation spectrum model takes the form:

$$\overline{H}_K(\nu) = H_K(\nu) + G_\infty \delta(\nu), \quad (13)$$

where: $\delta(\nu)$ denotes the Dirac delta function. Unbounded component $G_\infty\delta(\nu)$ of the relaxation spectrum $\overline{H}_K(\nu)$ (13) corresponds with the infinite relaxation time.

RELAXATION SPECTRUM APPROXIMATION

Suppose, a finite discrete experiment (stress relaxation test [13,17]) performed on the specimen of the material under investigation resulted in a set of measurements of the modulus $\overline{G}(t_i) = G(t_i) + z(t_i)$ at the sampling instants $t_i \geq 0$, $i = 1,\ldots,N$, where $z(t_i)$ is measurement noise. Optimal identification of the relaxation spectrum $H(\nu)$ in the class of models (13) consists of selecting within the given class of models defined by (13), (6) such a model, which ensures the best fit to the measurement results $\{\overline{G}(t_i)\}$. As a measure of the model accuracy the square approximation error is taken:

$$Q_N(\boldsymbol{g}_K) = \sum_{i=1}^{N} \left[\overline{G}(t_i) - \overline{G}_K(t_i)\right]^2, \quad (14)$$

where: $\boldsymbol{g}_K = [g_1 \ \ldots \ g_K \ \ G_\infty]^T$ is the parameter vector of model $\overline{H}_K(\nu)$ (13), (6) or equivalently of the model $\overline{G}_K(t)$ given by (12), (7). Superscript "T" indicates transpose. Using the vector-matrix notation $\overline{\boldsymbol{G}}_N = [\overline{G}(t_1) \ \ldots \ \overline{G}(t_N)]^T$ and:

$$\boldsymbol{\Psi}_{N,K} = \begin{bmatrix} \phi_1(t_1,\alpha) & \ldots & \phi_K(t_1,\alpha) & 1 \\ \vdots & \ddots & \vdots & \vdots \\ \phi_1(t_N,\alpha) & \ldots & \phi_K(t_N,\alpha) & 1 \end{bmatrix}, \quad (15)$$

the model quality index can be rewritten in the compact form as $Q_N(\boldsymbol{g}_K) = \left\|\overline{\boldsymbol{G}}_N - \boldsymbol{\Psi}_{N,K}\,\boldsymbol{g}_K\right\|^2$. Here and further $\|\cdot\|$ denotes the Euclidean norm in the spaces R^N and R^{K+1}. For physically realistic materials the relaxation spectrum $H(\nu)$ is nonnegative definite for any $\nu \geq 0$. Thus, the requirement that the respective model $H_K(\nu)$ (6) is also nonnegative for any $\nu \geq 0$ is natural. The basic functions $\tilde{h}_k(\nu,\alpha)$ (4) are nonnegative. Therefore, if the model parameters are such that $g_k \geq 0$ for any $k = 1,\ldots,K$, then the model $H_K(\nu)$ is also nonnegative definite function. Obviously, the restriction that the model parameters are nonnegative is sufficient, but not necessary condition for

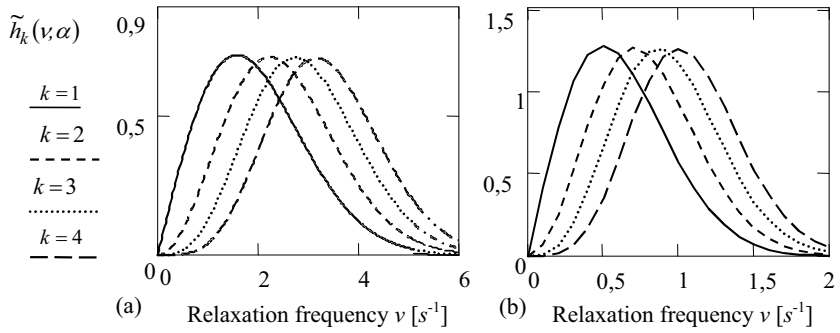

Fig. 1. The basis functions $\tilde{h}_k(\nu,\alpha)$ (4) for parameters: (a) $\alpha = 0{,}2\ [s^2]$ and (b) $\alpha = 2\ [s^2]$, $k = 1,2,3,4$

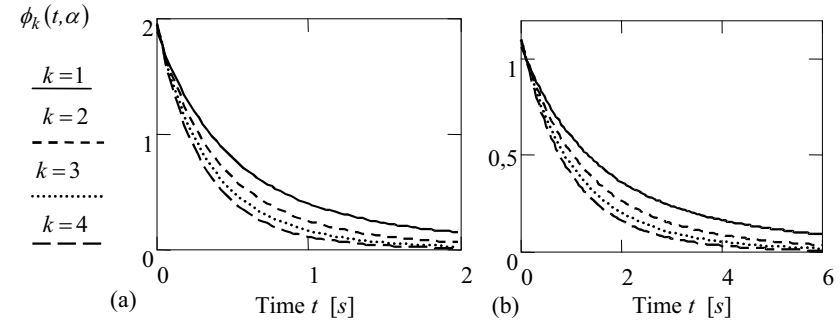

Fig. 2. The basis functions $\phi_k(t,\alpha)$, the parameters: (a) $\alpha = 0{,}2\ [s^2]$ and (b) $\alpha = 2\ [s^2]$, $k = 1,2,3,4$

the non-negation of the spectrum model. Now, the problem of the relaxation spectrum identification reduces to determining the vector of model parameters $\boldsymbol{g}_K$ minimizing the index $Q_N(\boldsymbol{g}_K)$ under the non-negation constraints $g_k \geq 0$ for $k=1,\ldots,K$. Thus, the linear least-squares optimization problem with inequality constraints of the form:

$$\min_{\boldsymbol{g}_K \geq \boldsymbol{0}_K} \quad \left\| \overline{\boldsymbol{G}}_N - \boldsymbol{\Psi}_{N,K}\, \boldsymbol{g}_K \right\|^2, \tag{16}$$

is stated for the optimal identification of the nonnegative relaxation spectrum. $\boldsymbol{0}_K$ is $(K+1)$-dimensional zero vector. The existence and properties of the solution of the above task depends on the properties of the matrix $\boldsymbol{\Psi}_{N,K}$ (15). Unfortunately, $\boldsymbol{\Psi}_{N,K}$ is usually rank- deficient. The linear-quadratic task (16) is still ill-conditioned and when the data are noisy, even small changes of the data $\overline{\boldsymbol{G}}_N$ would lead to arbitrarily large artefact in the optimal model parameters. Therefore, the numerical solution of finite dimensional problem (16) is fraught with the same difficulties that the original continuous ill-posed problem of numerical solution of the Fredholm equation (1). The standard minimization methods may fail. To stabilize the solution an additional "smoothing" constraint will be introduced in the next section.

OPTIMAL IDENTIFICATION PROBLEM

The fluctuations of the solution of optimization task (7) may be reduced by introducing an additional direct constraint on vector $\boldsymbol{g}_K$: $\|\boldsymbol{g}_K\| \leq \beta$, where a constant $0 < \beta < \left\| \overline{\boldsymbol{g}}_K^N \right\|$ estimates the „level of smoothness" assumed for the model parameters. Here $\overline{\boldsymbol{g}}_K^N$ is the normal (minimum Euclidean norm) solution of the original least-squares problem without constraints. In result the modified problem of optimal relaxation spectrum identification is obtained:

$$\min_{\boldsymbol{g}_K \geq \boldsymbol{0}_K} \quad \left\| \overline{\boldsymbol{G}}_N - \boldsymbol{\Psi}_{N,K}\, \boldsymbol{g}_K \right\|^2 \quad \textit{under constraint} \quad \|\boldsymbol{g}_K\|^2 \leq \beta^2. \tag{17}$$

By introducing a vector of prices (Lagrangian multipliers) $\boldsymbol{\lambda} \geq \boldsymbol{0}_K$ and a price $\gamma > 0$ we can define the Lagrangian for the optimization task (17):

$$L(\boldsymbol{g}_K, \boldsymbol{\lambda}, \gamma) = \left\| \overline{\boldsymbol{G}}_N - \boldsymbol{\Psi}_{N,K}\, \boldsymbol{g}_K \right\|^2 - \boldsymbol{\lambda}^T \boldsymbol{g}_K + \gamma\left(\boldsymbol{g}_K{}^T \boldsymbol{g}_K - \beta^2 \right). \tag{18}$$

The prices $\boldsymbol{\lambda}$ aim at providing a fulfillment of the inequality in original optimization task (16). The multiplier $\gamma > 0$ is the price imposed to satisfy the "smoothness" constraint in modified task (17). In order to use the dual approach to solve the optimization task (17) note, that for an arbitrary $\boldsymbol{\lambda} \geq \boldsymbol{0}_K$, $\gamma > 0$ and $\beta > 0$ the Lagrangian $L(\boldsymbol{g}_K, \boldsymbol{\lambda}, \gamma)$ has unique minimum with respect to $\boldsymbol{g}_K$ given by:

$$\hat{\boldsymbol{g}}_K(\boldsymbol{\lambda}, \gamma) = \left[\boldsymbol{\Psi}_{N,K}{}^T \boldsymbol{\Psi}_{N,K} + \gamma\, \boldsymbol{I}_{K,K} \right]^{-1} \left[\boldsymbol{\Psi}_{N,K}{}^T\, \overline{\boldsymbol{G}}_N + \tfrac{1}{2}\boldsymbol{\lambda} \right], \tag{19}$$

where: $\boldsymbol{I}_{K,K}$ is $(K+1)\times(K+1)$ identity matrix. Thus, the dual function defined by:

$$L_D(\boldsymbol{\lambda}, \gamma) = \min_{\boldsymbol{g}_K \in R^{K+1}} L(\boldsymbol{g}_K, \boldsymbol{\lambda}, \gamma) = L(\hat{\boldsymbol{g}}_K(\boldsymbol{\lambda}, \gamma), \boldsymbol{\lambda}, \gamma) \tag{20}$$

can be expressed by convenient analytical formula:

$$L_D(\boldsymbol{\lambda}, \gamma) = \overline{\boldsymbol{G}}_N{}^T \overline{\boldsymbol{G}}_N - \gamma\beta^2 - \left[\boldsymbol{\Psi}_{N,K}{}^T\, \overline{\boldsymbol{G}}_N + \tfrac{1}{2}\boldsymbol{\lambda} \right]^T \boldsymbol{B} \left[\boldsymbol{\Psi}_{N,K}{}^T\, \overline{\boldsymbol{G}}_N + \tfrac{1}{2}\boldsymbol{\lambda} \right], \tag{21}$$

where: symmetric matrix $\boldsymbol{B} = \left[\boldsymbol{\Psi}_{N,K}{}^T \boldsymbol{\Psi}_{N,K} + \gamma\, \boldsymbol{I}_{K,K} \right]^{-1}$ is positive definite for any $\gamma > 0$. It is easy to check that the Hessian matrix of the dual function takes the form:

$$H(\boldsymbol{\lambda}, \gamma) = \begin{bmatrix} -\frac{1}{2}\boldsymbol{B} & \boldsymbol{B}\boldsymbol{B}\left[\boldsymbol{\Psi}_{N,K}{}^T\, \overline{\boldsymbol{G}}_N + \frac{1}{2}\boldsymbol{\lambda} \right] \\ \left[\boldsymbol{\Psi}_{N,K}{}^T\, \overline{\boldsymbol{G}}_N + \frac{1}{2}\boldsymbol{\lambda} \right]^T \boldsymbol{B}\boldsymbol{B} & -2\left[\boldsymbol{\Psi}_{N,K}{}^T\, \overline{\boldsymbol{G}}_N + \frac{1}{2}\boldsymbol{\lambda} \right]^T \boldsymbol{B}\boldsymbol{B}\boldsymbol{B}\left[\boldsymbol{\Psi}_{N,K}{}^T\, \overline{\boldsymbol{G}}_N + \frac{1}{2}\boldsymbol{\lambda} \right] \end{bmatrix}.$$

Since matrix $\boldsymbol{B}$ is positive definite, on the basis of the known result concerning the definiteness of block matrices [12; Theorem $I_{b''}$] the Hessian $H(\boldsymbol{\lambda}, \gamma)$ is negative definite for an arbitrary $\boldsymbol{\lambda} \geq \boldsymbol{0}_K$. Thus the dual function $L_D(\boldsymbol{\lambda}, \gamma)$ is strictly concave function of the arguments $(\boldsymbol{\lambda}, \gamma)$. Taking advantage of the above, it may be proved that the solution to the dual problem:

$$\max_{\boldsymbol{\lambda} \geq \boldsymbol{0}_K,\, \gamma \geq 0} \quad L_D(\boldsymbol{\lambda}, \gamma) = L_D\left(\hat{\boldsymbol{\lambda}}, \hat{\gamma} \right), \tag{22}$$

there exists and the optimal multiplier $\hat{\gamma} > 0$. It is well known that if the saddle point of the Lagrangian $L(\boldsymbol{g}_K, \boldsymbol{\lambda}, \gamma)$ (18) there exists, then the dual approach can be successfully applied to solve (17). In the case considered the existence of a saddle point to the Lagrangian follows immediately from [8; Theorem 1, (ii) and (iii)] due to the uniqueness of (19). Thus the vector $\hat{\boldsymbol{g}}_K\left(\hat{\boldsymbol{\lambda}}, \hat{\gamma} \right)$ is optimal solution of the optimal identification task (17).

IDENTIFICATION SCHEME

The calculation of the relaxation spectrum model involves the following steps.

1. Perform the experiment - stress relaxation test [17] - and record the measurements $\overline{G}(t_i)$ of the relaxation modulus at time instants $t_i \geq 0$, $i = 1,\ldots,N$.
2. Compute $\left\|\overline{\boldsymbol{g}}_K^N\right\|$ (for simple analytical formula see, e.g., [18, eq. (2.24) and the following]) and choose the constant $0 < \beta < \left\|\overline{\boldsymbol{g}}_K^N\right\|$.
3. Solve the dual problem (22) according to the chosen numerical procedure, and determine the multipliers $(\hat{\boldsymbol{\lambda}}, \hat{\gamma})$ maximizing $L_D(\hat{\boldsymbol{\lambda}}, \hat{\gamma})$.
4. Compute the vector $\hat{\boldsymbol{g}}_K = \hat{\boldsymbol{g}}_K(\hat{\boldsymbol{\lambda}}, \hat{\gamma})$ of optimal model parameters using formula (19).
5. Determine the model of the spectrum of relaxation $\hat{H}_K(\nu)$ according to (cf. (6)):
6.

$$\hat{H}_K(\nu) = \sum_{k=1}^{K} \hat{g}_k \, \tilde{h}_k(\nu,\alpha) . \tag{23}$$

Obviously, $\overline{H}_K(\nu) = \hat{H}_K(\nu) + G_\infty \delta(\nu)$ is the relaxation spectrum of the form (13).

Taking advantage of the basic functions $\tilde{h}_k(\nu,\alpha)$ definitions (4), (5) and the properties of the optimal identification problem (17) solution, it may be simply proved that for $\hat{H}_K(\nu)$ (23) the following sequence of estimations hold:

$$\max_{\nu \geq 0} \left|\hat{H}_K(\nu)\right| \leq \sum_{k=1}^{K} \hat{g}_k \max_{\nu \geq 0} \tilde{h}_k(\nu,\alpha) \leq 1{,}084 \sqrt[4]{\alpha}\sqrt{K+1} \sum_{k=1}^{K} \hat{g}_k \leq 1{,}084 \sqrt[4]{\alpha}\sqrt{K+1}\,\beta .$$

Thus the smoothness of the vector $\hat{\boldsymbol{g}}_K$ guarantees that the fluctuations of the respective spectrum of relaxation $\hat{H}_K(\nu)$ are also bounded.

SOME REMARKS

1. For numerical algebraic computations of the scheme the singular value decomposition SVD [15] of the matrices $\boldsymbol{\Psi}_{N,K}$ (15) or $\boldsymbol{\Psi}_{N,K}{}^T \boldsymbol{\Psi}_{N,K}$ may be used - for details see the previous papers [18, 19]. Efficient algorithms for SVD are available nowadays in the form of optimized numerical procedures in most commonly used contemporary computational packets.

2. The matrix $\boldsymbol{\Psi}_{K,K}$ (15) depends on the choice of the basis functions as well as the sampling instants $\{t_i\}$, however, does not depend in the experiment results. Thus, when the identification scheme is applied for successive samples of the same material, the SVD computations have not to be multiple repeated while the same measurement points $\{t_i\}$ are kept.

3. The relaxation modulus basis functions $\phi_k(t,\alpha)$ (9)-(11) are expressed using complementary error function. The function $erfc(t)$ is accessible practically in every computational packets either directly or by error function $erf(t) = 1 - erfc(t)$, where $erf(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-x^2} dx$.

4. In the scheme proposed the parameter $\alpha > 0$ is the time-scaling factor. The following rule holds: the lower the parameter α is, the greater are the relaxation frequencies (see Figures 1 and 2). By the optimal choice of the scaling factor, the best fit of the model to the experimental data can be achieved. However, in practice a simple rough rule for choosing α, based on the comparison of a few first functions $\phi_k(t,\alpha)$ for different values of α with the experimentally obtained function $\overline{G}(t)$ is quite enough. In the same manner the number K of the models (6) and (7) summands can be initially evaluated. Thus, the choice both of the number K as well as the parameter α must be done *a posteriori*, after the preliminary experiment data analysis.

NUMERICAL EXAMPLE

To illustrate our approach we consider viscoelastic material whose relaxation spectrum is described by the Gauss distribution:

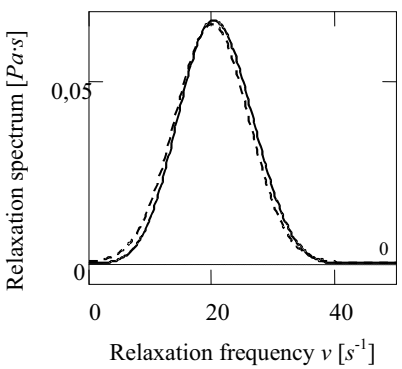

Fig. 3. Relaxation spectrum $H(\nu)$ (24) (dash line) and the nonnegative model $\hat{H}_K(\nu)$ (solid line)

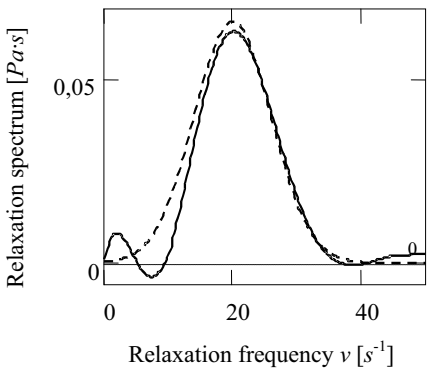

Fig. 4. Relaxation spectrum $H(\nu)$ (24) (dash line) and the least-squares optimal model $\tilde{H}_K(\nu)$ (solid line)

$$H(\nu)=\frac{1}{6\sqrt{2\pi}}e^{-(\nu-20)^2/72}. \quad (24)$$

The relaxation modulus corrupted by additive noises $z(t)$ of the uniform distribution in the interval $[-0{,}02;0{,}02]\,Pa$ has been sampled at $N=200$ sampling instants at the constant period $\Delta t=0{,}003\,[s]$. The parameters $K=8$ and $\alpha=0{,}003\,[s^2]$ are chosen according to the suggestions of Remark 4. Since $\|\bar{\boldsymbol{g}}_K^N\|=8{,}692$, the constant $\beta=0{,}2$ is assumed. Then the vector of the optimal model parameters $\hat{\boldsymbol{g}}_K$ is determined. The "real" relaxation spectrum $H(\nu)$ (24) and the resulting nonnegative definite model $\hat{H}_K(\nu)$ (23) are plotted in Figure 3. For comparison, the model $\tilde{H}_K(\nu)$ of the form (6) determined using the regularized least-squares method without constraints (for detailed description see, e.g., [18,19]) is given in Figure 4.

RELAXATION SPECTRUM OF THE REAL BIOLOGICAL MATERIAL

A cylindrical sample of 20 mm diameter and height was obtained from the root of sugar beet Janus variety [6]. The stress relaxation experiment performed by Gołacki and co-workers is described in details in [6]. The experiment was performed in the state of uniaxial deformation; i.e. the specimen examined underwent deformation in steel cylinder. The modeling of mechanical properties of this material in linear-viscoelastic regime is justified by the research results presented in a lot of works, for references see [6,18]. In the initial phase of the stress relaxation test the strain was imposed instantaneously, the sample was preconditioned at the 0,5 $[m{\cdot}s^{-1}]$ strain rate to the assumed strain. Next, during the second phase at constant strain, the corresponding time-varying force induced in the specimen, which decreases with time, was recorded during the time period $[0{,}5;96{,}2]$ seconds in 958 measurement points with the constant sampling period $\Delta t=0{,}1\,[s]$. The way how the experiment data has been preliminary proceeded is described in [18]. Next the proposed identification scheme was applied. Since $\|\bar{\boldsymbol{g}}_K^N\|=1{,}804E+5$, the constant

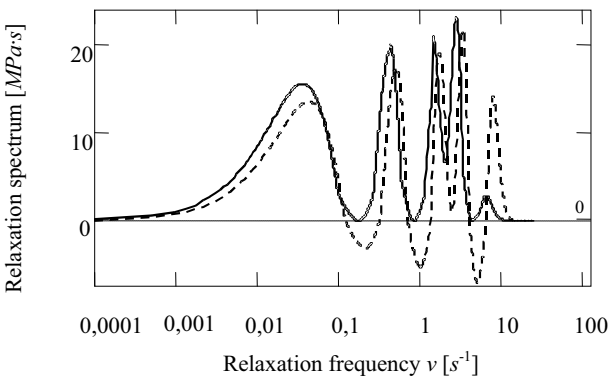

Fig. 5. The models $\hat{H}_K(\nu)$ (solid line) and $\tilde{H}_K(\nu)$ (dash line) of the relaxation spectrum of the sample of beet sugar root

$\beta=50$ is assumed. The parameters $K=8$ and $\alpha=98\,[s^2]$ were chosen. The resulting optimal nonnegative relaxation spectrum $\hat{H}_K(\nu)$ is plotted in Figure 5, where the relaxation spectrum model $\tilde{H}_K(\nu)$ of the class (6) optimal in the sense of regularized least-squares without constrains is also shown. The respective optimal long-term modulus are following: $\hat{G}_\infty=9{,}957\ MPa$ and $\tilde{G}_\infty=9{,}782\ MPa$.

CONCLUSIONS

The problem of the optimal least-squares approximation of the relaxation spectrum by finite series of nonnegative power basis functions is stated. The primary infinite dimensional problem of the relaxation spectrum identification is reduced to the static linear-quadratic programming task with inequality constraints. The dual approach is successfully applied to solved this task and resulting identification scheme is presented. The validity and effectiveness of the method is demonstrated both by numerical example as well as by through the computation of the relaxation spectrum of the specimen of selected biological material.

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O WYZNACZANIU NIEUJEMNEGO SPEKTRUM RELAKSACJI MATERIAŁÓW LEPKOSPRĘŻYSTYCH Z WYKORZYSTANIEM FUNKCJI RESZTKOWYCH BŁĘDU

Streszczenie. Spektrum relaksacji naprężeń charakteryzuje lepkosprężyste własności materiałów. Przedmiotem pracy jest problem wyznaczania spektrum częstotliwości relaksacji liniowych materiałów lepkosprężystych na podstawie dyskretnych, zakłóconych pomiarów modułu relaksacji zgromadzonych w teście relaksacji naprężeń. Przedstawiono nową metodę aproksymacji ciągłego spektrum relaksacji skończonym szeregiem nieujemnie określonych wykładniczych funkcji bazowych optymalnej w sensie najmniejszej sumy kwadratów. Pokazano, że funkcje bazowe odpowiedniego modelu modułu relaksacji dane są prostą regułą rekurencyjną i bazują na funkcji resztkowej błędu. Problem optymalnej identyfikacji nieujemnego spektrum relaksacji rozwiązano stosując podejście dualne. Przedstawiono schemat obliczeniowy algorytmu identyfikacji. Metodę zilustrowano przykładem numerycznym. Wyznaczono także spektrum relaksacji próbki buraka cukrowego badanego w stanie jednoosiowego odkształcenia przy zakłóconych pomiarach siły reakcji próbki.
Słowa kluczowe: lepkosprężystość, spektrum relaksacji naprężeń, identyfikacja, funkcje resztkowe błędu.