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IDENTIFICATION METHOD FOR THE STOCHASTIC INTRINSIC GROWTH MODELS WITH TIME-VARYING PARAMETERS

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Summary

The paper presents the parameter estimation method for stochastic intrinsic growth models with time-varying parameters. The method is based on the ideas of optimal control theory. The illustrations of the identification method show the proficiency of the presented ideas.

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AMS classification 2010: 92D25, 60H10, 93E12

1. Introduction

The communities of animals and plants are some examples of biological systems, where single-species population dynamics $X(t) \in \mathrm{R}$, $t \in [t_0, t_1]$, depends on environmental variability, internal transformations, human control factor, and etc. Moreover, it is hardly possible to isolate entirely one population from the rest. Different mathematical models have been proposed to describe the dynamics of biological populations (see Edelstein-Keshet, 2005, or Murray, 2006). One of the effective approach is to use continuous model presented by ordinary differential equations (ODE)

$$dX(t) = X(t)\,a(X(t),\theta)\,dt\,,\ X(t_0) = X_0\,, \tag{1.1}$$

where the intrinsic growth rate of the population can be written as an infinite power series for sufficiently smooth $a(\cdot,\cdot)$ as follows

$$a(X(t),\boldsymbol{\theta}) = \sum_{n=0}^{\infty} \theta_n X^n(t) = \theta_0 + \theta_1 X(t) + \theta_2 X^2(t) + \ldots, \tag{1.2}$$

with coefficients $\boldsymbol{\theta} = [\theta_0, \theta_1, \ldots]^T \subseteq \mathsf{R}$.

The model (1.1) presents the generalization of exponential growth model

$$dX(t) = \theta_1 X(t)\,dt\,,\ \ X(t_0) = X_0\,, \tag{1.3}$$

and logistic growth model

$$dX(t) = \left(\theta_1 X(t) + \theta_2 X^2(t)\right)dt\,,\ \ X(t_0) = X_0\,, \tag{1.4}$$

(where $\theta_1 = \rho$, $\rho \subseteq \mathsf{R}$, is the coefficient of the natural growth of the population, $\theta_2 = -\rho/K$ with $K \neq 0$, $K \subseteq \mathsf{R}$, is the carrying capacity of the environment for the species). The model (1.1) is more realistic to the true population dynamics. However, the natural growth and the carrying capacity are never constant and vary with time, model (1.1) has to be modified. Taking into account random disturbances, as it has been done by Filatova (2011), the single-species population dynamics can be written as a stochastic differential equation (SDE)

$$dX(t) = a(t, X(t), \boldsymbol{\theta}(t))\,dt + b(t, X(t), \boldsymbol{\theta}(t))\,dB(t)\,,\ X(t_0) = X_0\,, \tag{1.5}$$

where $\mathbf{a}:[t_0,t_1]\times\mathsf{R}\times\Theta\to\mathsf{R}$, $\mathbf{b}:[t_0,t_1]\times\mathsf{R}\times\Theta\to\mathsf{R}$ with Θ being a given metric space, which specifies the set of non-random values for the parameters $\boldsymbol{\theta}$, $\boldsymbol{\theta}(\cdot)$ is the unknown non-random vector of parameters, and $dB(t)$ is an increment of some stochastic process $B(t)$.

The model (1.5) will be call the stochastic intrinsic growth models with time-varying parameters. The goal is to present the estimation method for the parameters $\boldsymbol{\theta}$ of the model (1.5), taking into account some properties of the stochastic process $B(t)$.

2. Identification method for the time-varying parameters

2.1. Basic assumptions

For the simplicity in further reasoning we consider the SDE (1.5) limiting the family of stochastic processes $B(t)$ to one-dimensional ordinary Brownian motion (Bm):

$$B = \left\{ B(t), t \in \mathsf{R} \right\}, \tag{2.1}$$

which is a centered Gaussian process with stationary increments such that $\mathsf{E}\left[B(t)\right]=0$ with probability 1 and $\operatorname{var} B(t) = |t|$ ($\mathsf{E}[\cdot]$ denotes the expectation operator and $\operatorname{var}[\cdot]$ denotes the variance operator). In this case, we assume, that the solution $X(t)$, $t \in [t_0, t_1]$, of (1.5) exists and is unique in strong sense, and the vector of parameters $\boldsymbol{\theta}$ consists from r elements.

Now, let us denote the functions: $f : \mathsf{R}\times\mathsf{R}\times\mathsf{R}^r \to \mathsf{R}$, $\phi : \mathsf{R}\times\mathsf{R}\times\mathsf{R}^r \to \mathsf{R}$, $\psi_0 : \mathsf{R}\times\mathsf{R} \to \mathsf{R}$, $\psi_1 : \mathsf{R}\times\mathsf{R} \to \mathsf{R}^{k_1}$, $\psi_2 : \mathsf{R}\times\mathsf{R} \to \mathsf{R}^{k_2}$, $g_i : \mathsf{R}\times\mathsf{R} \to \mathsf{R}$ ($1 \le i \le l_1$), $\varphi_j : \mathsf{R}\times\mathsf{R}^r \to \mathsf{R}$ ($1 \le j \le l_2$) and formulate the identification task as follows

$$\mathbf{J}\left(X(\cdot), \boldsymbol{\theta}(\cdot)\right) = \inf_{\boldsymbol{\theta}\in\Theta} \left\{ \int_{t_0}^{t_1} f\left(t, X(t), \boldsymbol{\theta}(t)\right) dt + \psi_0\left(X(t_0), X(t_1)\right) \right\}, \tag{2.2}$$

($\inf(\cdot)$ is the greatest lower bound) subjected to

- object equation (1.5)

$$dX(t) = \phi\left(t, X(t), \boldsymbol{\theta}\right) dt, \ X(t_0) = X_0, \ a.e.\ t \in [t_0, t_1], \tag{2.3}$$

- constraints on all possible initial and terminal values of X

$$\psi_1\left(X\left(t_0\right), X\left(t_1\right)\right) \le 0\,; \tag{2.4}$$

$$\psi_2\left(X\left(t_0\right), X\left(t_1\right)\right) = 0\,; \tag{2.5}$$

- phase constraints

$$g_i\left(t, X\left(t\right)\right) \le 0, \quad \forall t \in \left[t_0, t_1\right], \; i = 1, \dots, l_1\,; \tag{2.6}$$

- constraints on parameters

$$\varphi_j\left(t, \boldsymbol{\theta}\left(t\right)\right) \le 0, \quad i = 1, \dots, l_2\,. \tag{2.7}$$

The problem of the parameters estimation of (1.5) in the formulation (2.2) - (2.7) is the optimal control task (see Milyutin et al., 2004).

There are many possibilities to solve the general optimal control problem (2.2) - (2.7). Since the solution of the object equation (1.5) is a stochastic process, it is reasonable to use stochastic principles as it was done by Hu et al., 2003. However, in our case we are not going to solve "pure" optimal control task. Since we consider a non-random vector of parameters, the SDE (1.5) can be converted to the ODE by means of moment equations (see Filatova et al, 2010). The next section presents the estimation principles.

2.2. Estimation principle

Let $m_1\left(t\right) = \mathsf{E}\left[X\left(t\right)\right]$ and $m_2\left(t\right) = \mathsf{E}\left[X^2\left(t\right)\right]$ be the first and second moments of stochastic process $X\left(t\right)$, $t \in \left[t_0, t_1\right]$, generated by the SDE (1.5). Denote a new state variable

$$\mathbf{y}\left(t\right) = \left[m_1\left(t\right), m_2\left(t\right)\right] \in \mathsf{R}^2,$$

where $\mathbf{y}\left(t_0\right) = \left[m_1\left(t_0\right), m_2\left(t_0\right)\right]$ ($m_1\left(t_0\right) = \mathsf{E}\left[X_0\right]$, $m_2\left(t_0\right) = \mathsf{E}\left[X_0^2\right]$), and describe object dynamics using a system of the ODEs

$$d\mathbf{y}\left(t\right) = \varphi\left(t, \mathbf{y}\left(t\right), \boldsymbol{\theta}\left(t\right)\right) dt \text{ a.e. } t \in \left[t_0, t_1\right]. \tag{2.8}$$

In this manner we have the possibility to use the principle maximum of Pontryagin, described by Milyutin and Osmolovskii, 1998, or Milyutin et al., 2004, to solve the parameter estimation problem. Now we introduce several definitions, which help to construct the estimation method.

Definition 2.1. Any $\boldsymbol{\theta}(\cdot)$ is called a feasible parameters vector $\boldsymbol{\theta}_f(\cdot)$, if

- $\boldsymbol{\theta}(\cdot) \in \mathbf{V}[t_0, t_1]$, where $\mathbf{V}[t_0, t_1] @ \{\boldsymbol{\theta} : [t_0, t_1] \to \Theta \,|\, \boldsymbol{\theta}(\cdot) \text{ is measurable}\}$;
- $\mathbf{y}(\cdot)$ is the unique solution of the system of the ODEs (2.8) under $\boldsymbol{\theta}(\cdot)$;
- the state constraints (2.4) and (2.5) are satisfied;
- $f(t, \mathbf{y}(t), \boldsymbol{\theta}(t))$ belongs to the set of Lebesgue measurable functions such that

$$\int_{t_0}^{t_1} \left| f(t, \mathbf{y}(t), \boldsymbol{\theta}(t)) \right| dt < \infty . \qquad \blacksquare$$

Definition 2.2. $\hat{\boldsymbol{\theta}}(\cdot)$ is called an optimal estimate of $\boldsymbol{\theta}(\cdot)$, if $\mathbf{J}\left(\hat{\mathbf{y}}(\cdot), \hat{\boldsymbol{\theta}}(\cdot)\right)$ is measurable and there exists $\varepsilon > 0$ such that for any $\mathbf{u}_f(\cdot)$ the following inequalities are fulfilled

$$\left\| \mathbf{y}(\cdot) - \hat{\mathbf{y}}(\cdot) \right\|_{\mathbf{C}([t_0, t_1], \mathrm{R}^2)} < \varepsilon ,$$

$$\mathbf{J}(\mathbf{y}(\cdot), \boldsymbol{\theta}(\cdot)) \geq \mathbf{J}\left(\hat{\mathbf{y}}(\cdot), \hat{\boldsymbol{\theta}}(\cdot)\right),$$

where $\mathbf{C}([t_0, t_1], \mathrm{R}^2)$ is the set of all continuous functions, $\hat{}$ denotes the estimate. $\blacksquare$

The definitions 2.1 and 2.2 allow us to propose the goal function (2.2) as follows

$$\mathbf{J}(\mathbf{y}(\cdot), \boldsymbol{\theta}(\cdot)) = \inf_{\boldsymbol{\theta} \in \Theta} \int_{t_0}^{t_1} f(t, \mathbf{y}(t), \boldsymbol{\theta}(t))\, dt ,$$

where

$$f(t, \mathbf{y}(t), \boldsymbol{\theta}(t)) = \left\| \varphi(t, \mathbf{y}, \boldsymbol{\theta}) - \varphi\left(t, \hat{\mathbf{y}}, \hat{\boldsymbol{\theta}}\right) \right\|_2^2 .$$

The phase constraints (2.6) and state constraints (2.7) can be defined on the basis of the properties of the stochastic process (see Shyryaev, 1998). As it was said before, the Pontryagin’s type maximum principle has to be used to find the solution to the estimation problem. In this case we introduce the Pontryagin’s function

$$\mathbf{H}\left(t,\mathbf{y}(t),\boldsymbol{\theta}(t),\psi(t)\right)=\psi(t)\varphi\left(t,\mathbf{y}(t),\boldsymbol{\theta}(t)\right)-\alpha_0 f\left(t,\mathbf{y}(t),\boldsymbol{\theta}(t)\right), \qquad (2.9)$$

where $\psi(t)\in\left(\mathrm{R}^2\right)'$ is an adjoint function of bounded variation ($\psi : [t_0,t_1]\to\mathrm{R}^2$ is an absolutely continues function), α_0 is a number.

The theorem below, based on Dubovitski-Milyutin method (see Milyutin et al., 2004), gives the possibility to find an optimal estimate $\hat{\boldsymbol{\theta}}(\cdot)$ of $\boldsymbol{\theta}(\cdot)$ for SDE (1.5).

Theorem 2.1. Let $\hat{\boldsymbol{\theta}}(\cdot)$ be an optimal estimate of $\boldsymbol{\theta}(\cdot)$ and $\left(\hat{\mathbf{y}}(\cdot),\hat{\boldsymbol{\theta}}(\cdot)\right)$ be an optimal pair ($\boldsymbol{\theta}(\cdot)\in\mathbf{L}^\infty\left([t_0,t_1],\mathrm{R}^2\right)$, $\mathbf{y}(\cdot)\in\mathbf{C}\left([t_0,t_1],\mathrm{R}^2\right)$). Then there exist a number α_0, a function of bounded variation $\psi(t)$ (which defines the measure $d\psi$), a function of bounded variation $\lambda(t)$ (which defines the measure $d\lambda$) such that the following conditions hold:

- nontriviality $|\alpha_0|+\|d\lambda\|>0$,
- nonnegativity $\alpha_0\geq 0$, $d\lambda\geq 0$,
- complementary slackness $d\lambda(t)g(t,\mathbf{y}(t))=0$,
- adjoint equation

$$-d\boldsymbol{\psi}(t)=\boldsymbol{\psi}(t)\varphi_{\mathrm{y}}\left(t,\hat{\mathbf{y}}(t),\hat{\boldsymbol{\theta}}(t)\right)-\alpha_0 f_{\mathrm{y}}\left(t,\hat{\mathbf{y}}(t),\hat{\boldsymbol{\theta}}(t)\right)-g_{\mathrm{y}}\left(t,\hat{\mathbf{y}}(t)\right)d\lambda, \qquad (2.10)$$

- transversality condition $\boldsymbol{\psi}(t_1)=0$,
- the local maximum condition

$$\psi(t)\varphi_{\boldsymbol{\theta}}\left(t,\hat{\mathbf{y}}(t),\hat{\boldsymbol{\theta}}(t)\right)-f_{\boldsymbol{\theta}}\left(t,\hat{\mathbf{y}}(t),\hat{\boldsymbol{\theta}}(t)\right)=0. \qquad (2.11)$$

■

The proof of the theorem 2.1 is not complicated. The general case of the proof can be found in Milyutin and Osmolovskii (1998) or in Filatova (2012).

3. Example

According to Atlantic State Marine Fisheries Commission (http://www.asmfc.org) as of 2010 the Atlantic herring was not threatened by overfishing. This fact can be explained by reduced number of herring natural predators among which were cod fish and salmon and in consequence by unlimited growth of herring population. The data (see Fig.1), which describe this population, give the possibility to use the stochastic intrinsic growth model as

$$dX(t) = \theta_1(t)\, X(t)\, dt + \theta_2(t)\, X(t)\, dB(t), \quad X(t_0) = X_0, \tag{3.1}$$

where $\theta(t) = \left[\theta_1(t), \theta_2(t)\right]^{\mathsf{T}}$ is a vector of unknown parameters.

Taking into account the martingale property of (2.1) and applying Ito formula to $\mathsf{E}\left[X(t)\right]$ and $\mathsf{E}\left[X^2(t)\right]$ we replace (3.1) by the following system of ODEs

$$\begin{cases} dm_1(t) = & \theta_1(t)\, m_1(t)\, dt, \\ dm_2(t) = & \left(2\theta_1(t) + \theta_2^2(t)\right) m_2(t)\, dt, \end{cases} \tag{3.2}$$

where $m_1(t_0) = \mathsf{E}\left[X_0\right]$, $m_2(t_0) = \mathsf{E}\left[X_0^2\right]$.

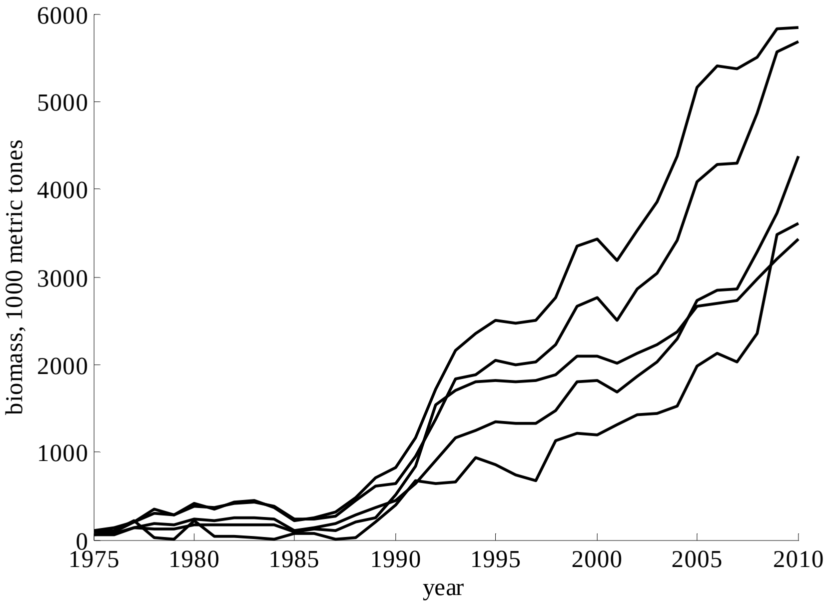

Fig. 1. Annual observations of North Atlantic herring population counted by five independent observatory

Denote $\hat{\varphi}\left(t,\hat{\mathbf{y}},\hat{\boldsymbol{\theta}}\right)=\left[\hat{\theta}_1(t)\hat{m}_1(t),\left(2\hat{\theta}_1(t)+\hat{\theta}_2^2(t)\right)\hat{m}_2(t)\right]^{\mathsf{T}}$ and $\varphi\left(t,\mathbf{y},\boldsymbol{\theta}\right)=\left[\varphi_1,\varphi_2\right]^{\mathsf{T}}$ (values of $\varphi\left(t,\mathbf{y},\boldsymbol{\theta}\right)$ can be found on the basis of the given data). The goal function (2.2) takes form

$$\mathbf{J}\left(\hat{\mathbf{y}}(\cdot),\hat{\boldsymbol{\theta}}(\cdot)\right)_{\min\hat{\boldsymbol{\theta}}}=\int_{t_0}^{t_1}\left[\left(\varphi_1-\hat{\theta}_1(t)\hat{m}_1(t)\right)^2+(\varphi_2-\left(2\hat{\theta}_1(t)+\hat{\theta}_2^2(t)\right)\hat{m}_2(t))^2\right]dt$$

The phase constraints (2.6) come from moment properties

$$g_1\left(t,\mathbf{y}(t)\right):m_1(t)-m_2(t)\le 0,$$
$$g_2\left(t,\mathbf{y}(t)\right):-m_2(t)\le 0,$$

and the constraints on the parameters (2.7) are not taken on the account. The Pontryagin function (2.9) can be presented as

$$\mathbf{H}\left(t,\hat{\mathbf{y}}(t),\hat{\boldsymbol{\theta}}(t),\psi(t)\right)=\psi_{m_1}(t)\hat{\theta}_1(t)\hat{m}_1(t)+\psi_{m_2}(t)\left(2\hat{\theta}_1(t)+\hat{\theta}_2^2(t)\right)\hat{m}_2(t)$$
$$-\alpha_0\left[\left(\varphi_1-\hat{\theta}_1(t)\hat{m}_1(t)\right)^2+\left(\varphi_2-\left(2\hat{\theta}_1(t)+\hat{\theta}_2^2(t)\right)\hat{m}_2(t)\right)^2\right],$$

where $\psi=\left[\psi_{m_1}(t),\psi_{m_2}(t)\right]^{\mathsf{T}}$.

In this case the conditions of the theorem 2.1 can be rewritten in following manner:

- nontriviality

$$|\alpha_0|+\|d\lambda_1\|+\|d\lambda_2\|>0, \tag{3.3}$$

- nonnegativity

$$\alpha_0\ge 0, d\lambda_1\ge 0, d\lambda_2\ge 0, \tag{3.4}$$

- complementary slackness

$$\begin{aligned}\left(\hat{m}_1(t)-\hat{m}_2(t)\right)d\lambda_1(t)&=0,\\ \hat{m}_2(t)\,d\lambda_2(t)&=0,\end{aligned} \tag{3.5}$$

- adjoint equation

$$
\begin{aligned}
-d\psi_{m_1}(t) = {} & \psi_{m_1}(t)\hat{\theta}_1(t)\,dt + 2\alpha_0\hat{\theta}_1(t)\left(\varphi_1 - \hat{m}_1(t)\right)dt - d\lambda_1, \\
-d\psi_{m_2}(t) = {} & \psi_{m_2}(t)\left(2\hat{\theta}_1(t) + \hat{\theta}_2^2(t)\right)dt - 2\alpha_0\Big[\varphi_2\left(2\hat{\theta}_1(t) + \hat{\theta}_2^2(t)\right) \\
& + \hat{m}_2(t)\left(4\hat{\theta}_1^2(t) + 4\hat{\theta}_1(t)\hat{\theta}_2^2(t) + \hat{\theta}_2^4(t)\right)\Big] + d\lambda_1 + d\lambda_2,
\end{aligned}
\tag{3.6}
$$

- transversality condition

$$
\begin{aligned}
\psi_{m_1}(t_1) &= 0, \\
\psi_{m_2}(t_1) &= 0,
\end{aligned}
\tag{3.7}
$$

- the local maximum condition

$$
\begin{aligned}
0 = {} & \psi_{m_1}(t)\hat{m}_1(t) + 2\psi_{m_2}(t)\hat{m}_2(t) - 2\alpha_0\Big(\hat{\theta}_1^2\hat{m}_1(t) - 2\varphi_2\hat{m}_2(t) \\
& + 4\hat{\theta}_1(t)\hat{m}_2^2(t) + 2\hat{\theta}_2^2(t)\hat{m}_2^2(t)\Big), \\
0 = {} & \psi_{m_2}(t)\hat{m}_2(t) - 4\alpha_0\left(-\varphi_2\hat{m}_2(t) + 2\hat{\theta}_1(t) + \hat{\theta}_2^2(t)\hat{m}_2^2(t)\right).
\end{aligned}
\tag{3.8}
$$

Numerical solution of (3.2) and (3.7) under conditions (3.4) – (3.6), (3.8), (3.9) of the theorem 2.1 with respect to $\theta_1(t)$ and $\theta_2(t)$ shows Fig.2. Finally, the model (3.1) takes form

$$
\begin{aligned}
dX(t) &= (0.0082 + 0.0019t)\,X(t)\,dt + 0.0256X(t)\,dB(t), \\
X(t_0) &= 500*10^3.
\end{aligned}
\tag{3.9}
$$

Conclusions

In this paper, the estimation method for stochastic differential equation with time-varying parameters was proposed. The same scheme can be used if one is interesting in parametric identification of a system of ordinary differential equations. The method is based on the ideas of optimal control theory, namely on the Dubovitski-Milyutin method. In the future, the numerical experiments are intended to take place in order to investigate the accuracy of the method.

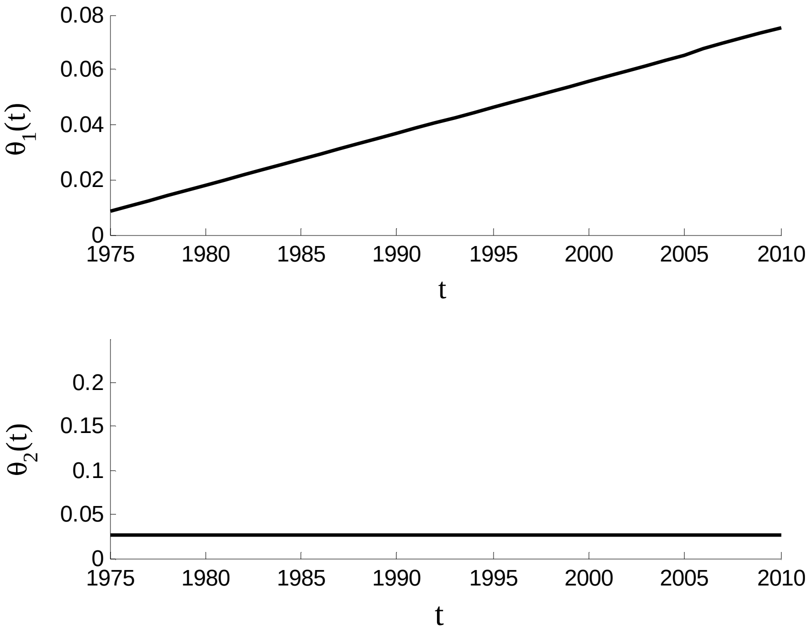

Fig. 2. Results of the identification

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