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On measurement point-independent identification of maxwell model of viscoelastic materials

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Summary. The problem of a weighted least-squares approximation of viscoelastic material by generalized Maxwell model is discussed when only the noise-corrupted time-measurements of the relaxation modulus are accessible for identification. To build a Maxwell model, which does not depend on sampling instants is a basic concern. It is shown that even when the true relaxation modulus description is completely unknown, the approximate optimal Maxwell model parameters can be derived from the measurement data sampled randomly according to appropriate randomization. The determined approximate model is a strongly consistent estimate of the requested model. An identification algorithm leading to the best model will be presented in the forthcoming paper, in which the convergence analysis will be also conducted. A motivating example is given.

Key words: viscoelasticity, relaxation modulus, Maxwell model, model identification

INTRODUCTION

Viscoelastic materials present a behaviour that implies dissipation and storage of mechanical energy. Viscoelastic models are used before all to modelling of different polymeric liquids and solids [3, 6, 14], concrete [1], soils [13], rubber [30], glass [5, 23], foods [2, 20, 22, 24]. Research studies conducted during the past few decades proved that these models are also an important tool for studying the behaviour of biological materials: wood [28], fruits, vegetables [8, 10, 11, 20], animals tissues [16], see also other papers cited therein.

Viscoelasticity of the materials manifests itself in different ways, such as gradual deformation of a sample of the material under constant stress (creep behaviour), and stress relaxation in the sample when it is subjected to a constant strain [6, 20, 29]. In general, viscoelasticity is a phenomenon associated with time variations in a material's response. In an attempt to describe some of the above effects mathematically several constitutive laws have been proposed which describe the stress–strain relations in terms of quantities like creep compliance, relaxation modulus, the storage and loss moduli and dynamic viscosity. Some of these constitutive laws have been developed with the aid of mechanical models consisting of combinations of springs and viscous dashpots. The Maxwell model is, perhaps, the most representative example of such models.

The classical Maxwell model is a viscoelastic body that stores energy like a linearized elastic spring and dissipates energy like a classical fluid dashpot. Within the past 40 years, advances in the Maxwell model study in the area of viscoelactic materials have been of three types. First the analysis of viscoelastic properties of such materials, e.g. elasticity, viscosity on the basis of Maxwell model, for example for polymers [3, 14], foods [2, 4, 20, 22, 24], biological materials [8, 9, 16], soda-lime-silica glass [5]. Next, the application of Maxwell model to compute other material functions such as the creep compliance, time-variable bulk and shear modulus or time-variable Poisson's ratio [27] or interconversion between linear viscoelastic material functions [19]. And finally, the development of computational tools for Maxwell model determining [21, 30]. This paper belongs to the latter group.

We often determine the parameters in a model by obtaining the „best-possible" fit to experimental data. The coefficients can be highly dependent on our way of measuring „best" [17]. Common choice of the model quality measure is the mean square approximation error, leading to a least-squares identification problem. When the identification index is fixed, the coefficients can be also highly dependent on the measurement data. To make the idea a little clear we give an example of the four-parameter Maxwell model determination of an confined cylindrical specimen of the beet sugar root.

To build a Maxwell model, which does not depend on sampling instants is a basic concern. We consider the problem of measurement point-independent approximation of a linear relaxation modulus of viscoelastic material within the class of discrete generalized Maxwell models when the integral weighted square error is to be minimized and the true material description is completely unknown. We show how the problem can be solved by introducing an appropriate randomization on the set of sampling instants at which the relaxation modulus of the material is measured. It is assumed that only the relaxation modulus measurements are accessible for identification. The idea of measurement point-independent identification was at first used for noiseless zero-memory system approximation by random choice of inputs in co-author paper [12].

IDENTIFICATION OF THE MAXWELL MODEL

MATERIAL

We consider a linear viscoelastic material subjected to small deformations for which the uniaxial, nonaging and izotropic stress-strain equation can be represented by a Boltzmann superposition integral [6]:

$$\sigma(t) = \int_{-\infty}^{t} G(t-\lambda)\dot{\varepsilon}(\lambda)\,d\lambda\,, \tag{1}$$

where: $\sigma(t)$ and $\varepsilon(t)$ denotes the stress and stain, respectively, and $G(t)$ is the linear time-dependent relaxation modulus. The modulus $G(t)$ is the stress, which is induced in the viscoelastic material described by equation (1) when the unit step strain $\varepsilon(t)$ is imposed.

By assumption, the exact mathematical description of the relaxation modulus $G(t)$ is completely unknown, but the value of $G(t)$ can be measured with a certain accuracy for any given value of the time $t \in \mathsf{T}$, where $\mathsf{T}=[0,T]$ and $0<T<\infty$ or $\mathsf{T}=R_+$; here $R_+=[0,\infty)$.

MAXWELL MODEL

The generalized discrete Maxwell model, which is used to describe the relaxation modulus $G(t)$, consists of a spring and n Maxwell units connected in parallel as illustrated in Figure 1. A Maxwell unit is a series arrangement of the Hooke and Newton's elements: an ideal spring in series with a dashpot. This model presents a relaxation of exponential type given by a finite Dirichlet-Prony series [29]:

$$G_M(t,\boldsymbol{g}) = \sum_{j=1}^{n} E_j\, e^{-v_j t} + E_\infty, \tag{2}$$

where: E_j, v_j and E_∞ represent the elastic modulus (relaxation strengths), relaxation frequencies and equilibrium modulus (long-term modulus), respectively. The vector of model (2) parameters is defined as:

$$\boldsymbol{g}=[E_1 \ldots E_n \quad v_1 \ldots v_n \quad E_\infty]^T. \tag{3}$$

The modulus E_j and the viscosity η_j associated with the *j–th* Maxwell mode (see Figure 1) determine the relaxation frequency $v_j=E_j/\eta_j$.

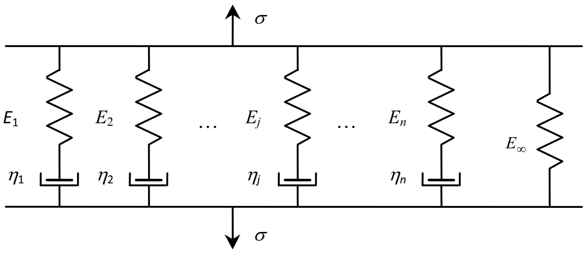

Fig. 1. Generalized discrete Maxwell model with additional elastic element E_∞ (Zeners' model)

It is not assumed that the real relaxation modulus $G(t)$ can be exactly represented within the chosen set of models (2), (3). The restriction that the model parameters are nonnegative and bounded must be given to satisfy the physical meaning, i.e. $\boldsymbol{g}\in\mathsf{G}$, where the admissible set of parameters G is compact subset of the space R_+^{2n+1}.

IDENTIFICATION OF THE MAXWELL MODEL

A classical manner of studying viscoelasticity is by two-phase stress relaxation test, where the time-dependent shear stress is studied for step increase in strain [20, 29]. Suppose, a certain stress relaxation test performed on the specimen of the material under investigation resulted in a set of measurements of the relaxation modulus $\bar{G}(t_i) = G(t_i) + z(t_i)$ at the sampling instants $t_i \geq 0$, $i=1,\ldots,N$, where $z(t_i)$ is measurement noise. Identification consists of selecting within the given class of models (2), (3) such a model, which ensures the best fit to the measurement results. As a measure of the model (2) accuracy the mean sum of squares is taken:

$$Q_N(\boldsymbol{g}) = \frac{1}{N}\sum_{i=1}^{N} \left[\bar{G}(t_i) - G_M(t_i,\boldsymbol{g})\right]^2. \tag{4}$$

This is the least-squares criterion for Maxwell model. Therefore the least-squares Maxwell model identification consists of determining the parameter $\boldsymbol{g}N$ minimizing the index (4) on the set G by solving the following optimization problem:

$$Q_N(\boldsymbol{g}_N) = \min_{\boldsymbol{g}\in G} Q_N(\boldsymbol{g}). \tag{5}$$

Exponential sum models are used frequently in applied research: time series in economics, biology, medicine, heat diffusion and diffusion of chemical compounds in engineering and agriculture, physical sciences and technology, see, e.g., [7, 18]. Fitting data to exponential sums is a very old problem, which has been studied for a long time. Several articles have appeared mainly to finding optimal least-squares exponential sum approximations to sampled data. Holmström and Petersson [15] have reviewed known algorithms in much detail.

The results of identification, both the model parameters and the resulting relaxation modulus are (strongly) dependent on the measurement data, in particular of the sampling instants t_i. This is best illustrated by an example.

EXPERIMENT AND MOTIVATING EXAMPLE

A cylindrical sample of 20 mm diameter and height was obtained from the root of sugar beet Janus variety [9]. During the two-phase stress relaxation test performed by Gołacki and co-workers at the University of Life Sciences in Lublin [9], in the first initial phase the strain was imposed instantaneously, the sample was preconditioned at the 0.5 $m \cdot s^{-1}$ strain rate to the maximum strain. Next, during the second phase at constant strain the corresponding time-varying force induced in the specimen was recorded during the time period [0,100] seconds in 40000 measurement points with the constant sampling period $\Delta t = 0.0025$s. The experiment was performed in the state of uniaxial deformation; i.e. the specimen examined underwent deformation in steel cylinder (for details see, for example, [9]). The modelling of mechanical properties of this material in linear-viscoelastic regime is justified by the research results presented in a lot of works, for example [8]. For initial filtering of the force measurement data Savitzky-Golay method has been used. Next, the respective relaxation modulus measurements were computed using a simple modification of the well-known Zapas and Craft [29] rule:

$$\bar{G}(t) = \bar{F}\left(t + t_{m,real}/2\right)/(\varepsilon_0 p) \quad for \quad t \geq t_{m,real}/2,$$

where: $t_{m,real}$ is the real time under which the induced force $\bar{F}(t)$ take the maximum value, ε_0 is the constant strain kept during the second phase of the test and p is the cross-section of the sample.

Table 1. Maxwell model (6) parameters and the values of identification index $Q_N(\boldsymbol{g}_N)$; equidistant-experiment

N	$Q_N(\boldsymbol{g}_N)$	Model parameters			
		$E_{1,N}$ [MPa]	$E_{2,N}$ [MPa]	$v_{1,N}$ [s^{-1}]	$v_{2,N}$ [s^{-1}]
15	0,0025	10,563	3,9242	8,6049E-4	1,7521
20	0,0011	10,5135	3,8923	7,8748E-4	0,6296
25	0,0021	10,5368	3,8794	8,1408E-4	0,7776
40	0,0025	10,5372	3,9074	8,0998E-4	1,171
50	0,0029	10,5515	3,9133	8,3473E-4	1,4523
75	0,0032	10,5604	3,9521	8,4544E-4	2,1065
100	0,0037	10,5596	3,9946	8,4957E-4	2,6723
150	0,004	10,567	4,1043	8,5423E-4	4,2195
200	0,0044	10,5664	4,2017	8,5543E-4	5,4635
250	0,0042	10,5721	4,2935	8,6228E-4	6,7021
300	0,0047	10,5716	4,438	8,6447E-4	8,4651
350	0,0047	10,5733	4,483	8,6492E-4	9,0439
400	0,0046	10,5717	4,6187	8,6186E-4	10,6006
500	0,0046	10,5746	4,8924	8,6621E-4	13,7081
600	0,0048	10,5747	5,0389	8,6689E-4	15,2986

For N from the set N = {15,20,25,40,50,75,100,150,200 ,250,300,350,400,500,600} equidistant sampling points t_i have been taken in time interval [0,95] seconds, successively, and the respective relaxation modulus measurements have been selected from the whole set of measurement data. Next, the Levenberg-Marquardt optimization procedure was applied to solve the optimization task (5) and the four parameter Maxwell models:

$$G_M(t) = E_1 e^{-v_1 t} + E_2 e^{-v_2 t}, \tag{6}$$

where the elastic modulus E_i and the relaxation frequencies v_i, $i = 1,2$, were determined for each N. The results of the identification, i.e. the optimal model parameters and the optimal values of the empirical index $Q_N(\boldsymbol{g})$ are given in Table 1.

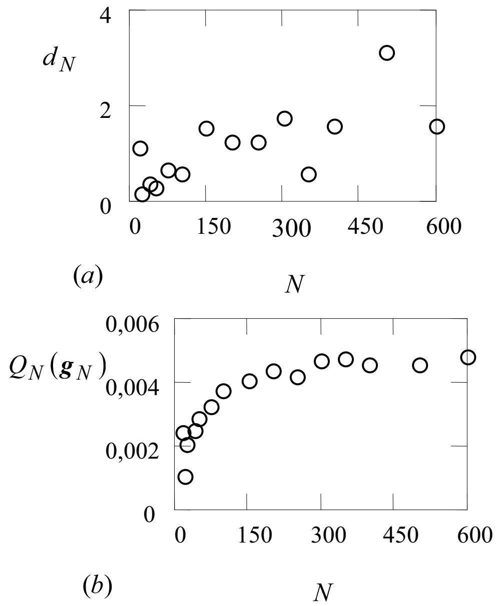

Fig. 2. (a) The distance d_N between the two successive Maxwell model parameters $\boldsymbol{g}_N$ and (b) the identification index $Q_N(\boldsymbol{g}_N)$ as a function of the number of measurements N; equidistant-experiment

To illustrate the convergence of the Maxwell model parameters in Figure 2(a) the distance $d_N = \|\boldsymbol{g}_N - \boldsymbol{g}_{[N]}\|_2$, where $[N]$ is a direct predecessor of N in the set N, is shown as a function of N; here $\|\cdot\|_2$ denotes the Euclidean norm in the space R^{2n+1}. The course of the model quality index as a function of N is illustrated in Fig. 2(b). The relaxation modulus computed according to the best 'equidistant' Maxwell models $G_M(t,\boldsymbol{g}_N)$ are plotted in Figure 3 for a few values of the number of measurements, where the measurements $\bar{G}(t_i)$ are also marked. However, the models $G_M(t,\boldsymbol{g}_N)$ does not differ significantly (see Figure 3), the model parameters differ essentially - compare Figure 2(a) and Table 1

The above example illustrates that the Maxwell model parameters will be highly dependent on the measurement data, if the sampling instants t_i are inappropriately chosen. This is a crucial point of the problem. Loosely speaking, the problem is, whether the identification procedure will yield a Maxwell model parameters which are asymptotically (when the number of measurements tends to infinity) independent on the particular sampling instants. The issue involves aspects on whether the data set (i.e. the experimental conditions) is informative enough to guarantee this convergence result. We show, that this problem can be satisfactorily solved by introducing a simple randomization on the sampling times set.

METHODS AND RESULTS

OPTIMAL APPROXIMATION OF THE MAXWELL MODEL

As a measure of the model (2), (3) accuracy the global approximation error of the form:

$$Q(\boldsymbol{g}) = \int_{T} \left[G(t) - G_M(t, \boldsymbol{g}) \right]^2 \rho(t)\, dt, \tag{7}$$

where a chosen weighting function $\rho(t) \geq 0$ is a density on T, i.e., $\int_T \rho(t)dt = 1$, can be taken. Thus, the problem of the real relaxation modulus $G(t)$ optimal approximation within the class of Maxwell models reduces, obviously, to determining the parameter $\boldsymbol{g}^*$ minimizing the index $Q(\boldsymbol{g})$ on the set of admissible parameters G, i.e. takes the form:

$$\boldsymbol{g}^* = arg \min_{\boldsymbol{g} \in G} Q(\boldsymbol{g}), \tag{8}$$

where $arg\,min_{g \in G} Q(\boldsymbol{g})$ denotes the vector $\boldsymbol{g}$ that minimizes $Q(\boldsymbol{g})$ on the set G. Note, that the empirical index $Q_N(\boldsymbol{g})$ (4) is obtained by the replacement of the integral in $Q(\boldsymbol{g})$ with the finite mean sum of squares.

MATHEMATICAL BACKGROUND AND ASSUMPTIONS

Let $T_1, \ldots, T_N$ are independent random variables with a common probability density function $\rho(t)$ whose support is T. Let $G_i = G(T_i)$ be the corresponding relaxation modulus, $i = 1, \ldots, N$, and let $\overline{G}_i = G_i + Z_i = G(T_i) + Z_i$ denote

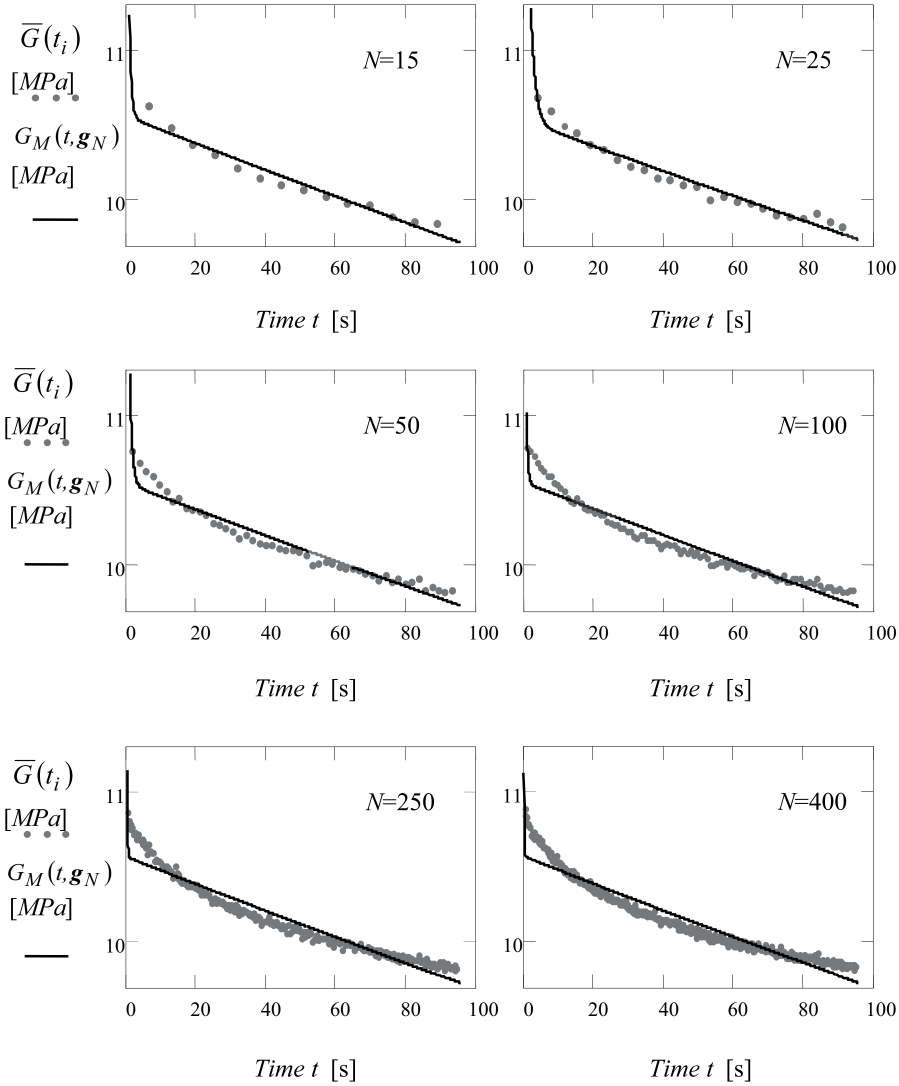

Fig. 3. The relaxation modulus measurements $\overline{G}(t_i)$ (points) and the approximate Maxwell models $G_M(t, \boldsymbol{g}_N)$ (solid line); equidistant-experiment

their measurements obtained in a certain stress relaxation test performed on the specimen of the material under investigation. Here Z_i are additive measurement noises.

We take the following assumptions, which seems to be quite natural in the context of relaxation modulus approximation task.

- **Assumption 1.** *The relaxation modulus* $G(t)$ *is bounded on* T, *i.e.* $sup_{t\in\mathsf{T}}G(t)\leq M<\infty$
- **Assumption 2.** *The set of admissible model parameters* G *is compact in the space* R_+^{2n+1}.
- **Assumption 3.** *The measurement noises* Z_i *are bounded, i.e.* $|Z_i|\leq\delta<\infty$ *for* $i=1,\ldots,N$.
- **Assumption 4.** $\{Z_i\}$ *is a time-independent sequence of independent identically distributed* (*i.i.d.*) *random variables with zero mean and a common finite variance* σ^2: $E[Z_i]=0$ *and* $E[Z_i]=\sigma^2<\infty$.

Note that the assumption 1 is satisfied, in particular, if $G(0)<\infty$ and the weak energy dissipation principle is satisfied – for details see, for example [25]. Obviously, from assumption 4 if follows that $E[G(T_i)+Z_i-G_M(T_i,\boldsymbol{g})]=Q(\boldsymbol{g})+\sigma$. Taking into account the Maxwell model equations (2), (3) set-up we see that the following properties hold.

- **Property 1.** $G_M(t,\boldsymbol{g})$ *is continuous and differentiable with respect to* $\boldsymbol{g}$ *for any* $t\in\mathsf{T}$.
- **Property 2.** $sup_{t\in\mathsf{T},g\in\mathsf{G}}\|\nabla\boldsymbol{g}G_M(t,\boldsymbol{g})\|_2<\infty$ *for an arbitrary compact subset* G *of* R_+^{2n+1}.
- **Property 3.** $sup_{t\in\mathsf{T},g\in\mathsf{G}}G_M(t,\boldsymbol{g})<\infty$ *for an arbitrary compact subset* G *of the space* R_+^{2n+1}.

Notice that, since in view of Property 1 the quality indices $Q(\boldsymbol{g})$ and $Q_N(\boldsymbol{g})$ are continuous with respect to $\boldsymbol{g}$, then if the set G is compact in the space R_+^{2n+1}, the solutions of the optimal approximation tasks (5) and (8) there exist, on the basis of the well-known Weierstrass's theorem which asserts the existence of continuous function extrema on compact sets.

ASYMPTOTIC PROPERTIES OF THE OPTIMAL MODEL

Now we wish to investigate the stochastic-type asymptotic properties of the Maxwell model approximation tasks (5) and (8). When studying these issues, the following proposition is instrumental.

- **Proposition 1.** *When the relaxation modulus measurements are corrupted by additive noise and the Assumptions* 1-4 *are satisfied, then*:

$$\sup_{g\in\mathcal{G}}\left|Q(\boldsymbol{g})+\sigma^2-Q_N(\boldsymbol{g})\right|\to 0 \quad w.p.1 \quad as \quad N\to\infty, \tag{9}$$

*where w.p.*1 *means "with probability one".*

The proof follows immediately from Property 2 in [12]. To verify this claim we need only note that the above Properties 1 and 2 guarantee that the assumptions A2 and A3 in [12] are satisfied. Next, the Assumption 2 is equivalent to A1, the Assumption 4 is equivalent to A5 ibidem, and due to Assumption 1 and Property 3 the assumption A4 ibidem holds.

Proposition 1 enables us to relate the Maxwell model parameter $\boldsymbol{g}_N$ solving the optimal approximation task (5) for empirical index $Q_N(\boldsymbol{g})$ to the parameter $\boldsymbol{g}^*$ minimizing the deterministic function $Q_N(\boldsymbol{g})$ in (8). Namely, from the uniform in $\boldsymbol{g}\in\mathsf{G}$ convergence of the index $Q_N(\boldsymbol{g})$ in (9) we conclude immediately the following.

- **Proposition 2.** *Assume that Assumptions 1-4 are in force,* $T_1,\ldots,T_N$ *being independently, at random selected from* T, *each according to probability distributions with density* $\rho(t)$. *Then for the additive noise corrupted relaxation modulus measurements*:

$$\boldsymbol{g}_N\to\boldsymbol{g}^* \qquad w.p.1 \quad as \quad N\to\infty \tag{10}$$

and for all $t\in\mathsf{T}$:

$$G_M\left(t,\boldsymbol{g}_N\right)\to G_M\left(t,\boldsymbol{g}^*\right) \qquad w.p.1 \quad as \quad N\to\infty.$$

Thus, under the taken assumptions the Maxwell model parameter $\boldsymbol{g}_N$ is strongly consistent estimate of the parameter $\boldsymbol{g}^*$. Moreover, since the Maxwell model $G_M(t,\boldsymbol{g})$ is Lipschitz on G uniformly in $t\in\mathsf{T}$ (the above is guaranteed by Property 2), then the almost sure convergence of $\boldsymbol{g}_N$ to the respective parameter $\boldsymbol{g}^*$ in (10) implies that:

$$\sup_{t\in T}\left|G_M\left(t,\boldsymbol{g}_N\right)-G_M\left(t,\boldsymbol{g}^*\right)\right|\to 0 \qquad w.p.1 \quad as \quad N\to\infty,$$

i.e., $G_M(t,\boldsymbol{g}_N)$ is in the case considered a strongly uniformly consistent estimate of the best model $G_M(t,\boldsymbol{g}^*)$.

CONCLUSIONS

Summarizing, when the Assumptions 1-4 are satisfied, the arbitrarily precise approximation of the optimal Maxwell model (with the parameter $\boldsymbol{g}^*$) can be obtained (almost everywhere) as the number of measurements N grows large, despite the fact that the real description of the relaxation modulus is completely unknown. Thus, when the set $\{t_i\}$ is open to manipulation during the data collection, it is an important experiment design issue to take an appropriate sampling instants. We shall comment on how to do this in the forthcoming paper [26], where the complete identification algorithm providing the strongly consistent estimate of the optimal model is given. The stochastic-type convergence analysis is also performed in [26] and the rate of convergence is discussed for the case when the measurements are perfect or corrupted by additive noises.

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O NIEZALEŻNEJ OD PUNKTÓW POMIAROWYCH IDENTYFIKACJI MODELU MAXWELLA MATERIAŁÓW LEPKOSPRĘŻYSTYCH

S t r e s z c z e n i e . Rezultat identyfikacji, czyli wyznaczony model zależy zarówno od przyjętej klasy modeli oraz przyjętego wskaźnika jakości modelu, jak i konkretnych danych pomiarowych. W pracy rozważa się problem optymalnej w sensie najmniejszej sumy kwadratów aproksymacji modułu relaksacji materiałów liniowo lepkosprężystych uogólnionym modelem Maxwella na podstawie dyskretnych, zakłóconych pomiarów modułu relaksacji zgromadzonych w teście relaksacji naprężeń. Zakłada się, że opis rzeczywistego modułu relaksacji jest całkowicie nieznany. Pokazano, że wprowadzając odpowiednia randomizację punktów pomiarowych można uzyskać model Maxwella asymptotycznie niezależny od punktów pomiarowych. Wyznaczony model jest silnie zgodnym estymatorem modelu optymalnego w sensie całkowego kwadratowego wskaźnika jakości niezależnego od punktów pomiarowych.

Odpowiedni algorytm identyfikacji będzie przedmiotem kolejnej pracy, w której przeprowadzona zostanie także analiza zbieżności modelu.

Słowa kluczowe: lepkosprężystość, moduł relaksacji, model Maxwella, identyfikacja modelu.