Mathematical model of deformation of railway sleeper track structure with the step change of stiffness on the elastic winkler foundation of the constant stiffness

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Summary. We considered the formulation of the problem of constructing a mathematical model of the deformation of the railway sleeper track structure with the step change of stiffness on the elastic Winkler foundation. The rail is represented as a beam of variable cross-section. To determine the displacements and angles of rotation of cross sections, we used Laplace transform method of equation and delta at the joint. As a result of simulation, we obtained the forms for moving semi-infinite beams (stacked rails) laying on the elastic foundation, which can be used for arbitrary values of stiffness "C" of the elastic foundation. We also proposed a simplified alternate variant of solution using the method of "small" parameter in case if the stiffness characteristics of two adjacent rail sections slightly differ.
Key words: rolling stock, rail, linear stiffness, deflection and rotation angle of the cross-section at the joint, stiffness of the rail foundation, Winkler foundation.

INTRODUCTION

An effective and long-term use of the equipment and the infrastructure, transport systems especially the railway transport. It can significantly reduce the maintenance costs of material flows moving through these systems [13, 14, 18, 19]. One of the methods to reduce costs is to prolong the term of the rail work due to their constant permutation in the process of deterioration from more loaded onto less loaded routes [12, 17]. But in any case, such a movement involves the necessity of joining the rails of various grades, which leads to a drastic change in stiffness of the rail linear filament at the joint. This causes the significant vertical dynamic forces and, accordingly, decreases the velocity of railway vehicles. To avoid this phenomenon it is necessary to define the parameters of the deformation of the rail at the joint stiffness and the range of the rail foundation.

RESULTS OF RESEARCH

The aim of the study is to determine the interaction between the wheels of railway vehicles and the rail with graded stiffness laying on the Winkler foundation.

To simplify the construction of the model we consider only one rail consisting of different types of rails and respectively different linear stiffness. The load from wheels of the railway vehicles is transferred to the rails and through ties to ballast which can be represented as Winkler foundation.

To describe the behaviour of the rail as the beam of variable linear stiffness, laying on the Winkler foundation, it is possible to use the differential equation that has the form [1, 7, 10, 16]:

$$EI_2\frac{d^4U^2(x)}{dx^4}+\left[C_{21}+(C_{22}-C_{21})\_\!\left|^{-}(x-x_k)\right.\right]\cdot U^2(x)=$$
$$=-\sum_{i=0}^{n+1}R_i\delta(x-x_i). \qquad (1)$$

The solution of the equation (1) with the classical approach is based on the dissection of the beam at the point x_k (the point of connection of two parts of the beam of different stiffness) and subsequent calculating of two semi-infinite beams laying on an elastic foundation of the constant stiffness (Fig. 1).

Consequent joining is at the same section $x = x_k$. When joining two sections of beams, in the section x_k, the compatibility of displacements and rotation angles of beams at the joint, should be observed, that is, the fulfilment of conditions:

$$\left.\begin{aligned} U_{21}(x_k) &= U_{22}(x_k), \\ \frac{dU_{21}}{dx}(x_k) &= \frac{dU_{22}}{dx}(x_k). \end{aligned}\right\} \qquad (2)$$

To obtain the equation of bending of the beam (rail) laying on the elastic foundation (ballast) we use the equation of the initial parameters for each side.

To solve the equation for each side of the beam we use the method of integral transformation of Laplace's equation [3, 5, 10, 16]:

$$\frac{d^4U}{dx_4}+4\alpha^4U=\frac{P_i}{E_1I_1}\cdot\delta(x-x_i)\,, \qquad (3)$$

$$\frac{d^4U}{dx_4}+4\alpha^4U=\frac{P_i}{E_2I_2}\cdot\delta(x_i-x)\,, \qquad (4)$$

where: $\alpha^4=\frac{1}{4}\frac{K_2}{EI_2}$, K_2 – stiffness of the elastic foundation (N/m^2), E – the coefficient of the elasticity (N/m^2), $EI_{(1,2)}$ – stiffness of sides 1 and 2 of the rail beam (N/m^2), $\delta(x_i-x)$ – the delta function, defined at the point x_i.

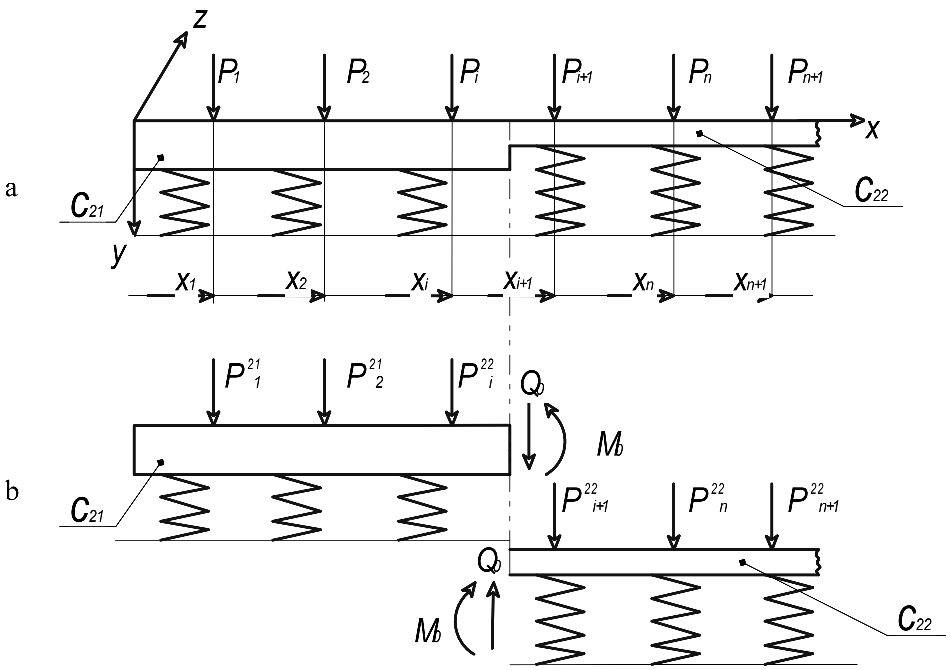

Fig. 1. Scheme of variable section beams on the elastic foundation:
a – calculation scheme of the beam with a step change of the linear stiffness C_{21}, C_{22} on the elastic foundation;
b – calculation schemes of beams with the constant linear stiffness on the elastic foundation C_{21}, C_{22}

If the coordinate system passes through a cross-section at the point x, from two semi-infinite beams (two rail sections of different stiffness), only one beam, which has less EI stiffness, can be viewed. This simplification is possible because ultimately it is necessary to know the movement of this side of the rail (beam) to choose the diagram of the rail track structure, which will provide the necessary stiffness of the bed and smooth bending of both sides of the beam, eliminate the appearance of additional dynamic loads from the wheel to the rail. According to fig. 1 that is the side of the beam with the stiffness C_{22}. Therefore, it is sufficient to consider only the equation (4).

After using the direct Laplace's equation [16]:

$$F(x)=\int_0^\infty e^{-st}U(t)dt\,. \tag{5}$$

We obtain:

$$S^4F(s)-U(0)S^3-U'(0)S^2-U''(0)S^2-4\alpha^4F(s)= \\ =\sum_i \frac{P_i}{EI}e^{-sx_i}. \tag{6}$$

After some equations:

$$F(s)=U(0)\frac{S^3}{S^4+4\alpha^4}+U'(0)\frac{S^2}{S^4+4\alpha^4}+ \\ +U''(0)\frac{S^2}{S^4+4\alpha^4}+U'''(0)\frac{1}{S^4+4\alpha^4}+\sum_i \frac{P_i}{EI}e^{-sx_i}. \tag{7}$$

To obtain the equation of the bending of the beam $U(x)$, laying on the elastic Winkler foundation it is necessary to make the inverse Laplace transformation by the formula:

$$U(x)=\lim_{a\to\infty}\frac{1}{2\pi i}\cdot\int_{a-ib}^{a+ib} e^{xs}F(s)ds\,. \tag{8}$$

Then obtain:

$$U(x)=U(0)\cos\alpha x\cdot ch\alpha x+ \\ +U'(0)\frac{1}{2\alpha}\left[sh\alpha x\cdot\cos\alpha x+ch\alpha x\cdot\sin\alpha x\right]+ \\ U''(0)\frac{1}{2\alpha^2}sh\alpha x\cdot\sin\alpha x+ \\ +U'''\frac{1}{4\alpha^3}\left[ch\alpha x\cdot\sin\alpha x+sh\alpha x\cdot\cos\alpha x\right]+ \\ +\frac{1}{4\alpha^3 EI}\sum_i P_i\begin{bmatrix} ch\alpha(x_i-x)\cdot\sin\alpha(x_i-x)- \\ -sh\alpha(x_i-x)\cdot\cos\alpha(x_i-x)\end{bmatrix}. \tag{9}$$

To simplify the function (9) we make the following change:

$$\left.\begin{aligned} & ch\alpha x\cdot\cos\alpha x=B_1(x), \\ & \frac{1}{2\alpha}(ch\alpha x\cdot\sin\alpha x+sh\alpha x\cdot\cos\alpha x)=B_2(x), \\ & \frac{1}{2\alpha^2}(sh\alpha x\cdot\sin\alpha x)=B_3(x), \\ & \frac{1}{4\alpha^3}(ch\alpha x\cdot\sin\alpha x-sh\alpha x\cdot\cos\alpha x)=B_4(x).\end{aligned}\right\} \tag{10}$$

Then it can be written as:

$$U(x)=U(0)\cdot B_1(x)+U'(0)\cdot B_2(x)+U''(0)\cdot B_3(x)+ \\ +U'''(0)\cdot B_4(x)+\frac{1}{EI}\sum_i P_iB_4(x_i-x), \tag{11}$$

at that:

$$B_4(x_i-x)=\begin{Bmatrix} B_4(x_i-x) \text{ if } x_i\ge x, \\ 0 \qquad \text{if } x_i\le x.\end{Bmatrix} \tag{12}$$

Using the coupling between the shear forces and moments at the point of connection of beams (rails) of different stiffness, as well as derivatives of the displacement of the beam it allows determining the above-mentioned moments and forces:

$$M(x)=-EI\frac{d^2U(x)}{dx^2}\to M(0)=-EI\cdot U''(0), \\ Q(x)=-EI\frac{d^3U(x)}{dx^3}\to Q(0)=-EI\cdot U''(0). \tag{13}$$

Using the equation (13) the expression for bending of the beam (11) can be transformed to the form:

$$U(x)=U(0)\cdot B_1(x)+U'(0)\cdot B_2(x)+ \\ +\frac{1}{EI}\left[-M(0)\cdot B_3(x)-Q(0)\cdot B_4(x)+\sum_i P_iB_4(x_i-x)\right]. \tag{14}$$

Arbitrary constants

$U(0)$, $U'(0)$, $M(0)$, $Q(0)$, in the equation (14) can be determined from the boundary conditions.

As stated above, if we cut the beam of the variable stiffness at the point of connection of the semi-infinite stiffness C_{21} and C_{22}, i.e., at the cross section $x = x_k$ (Fig. 1), and laying on the elastic foundation of the constant stiffness, for calculating each of these beam it is possible to use the equation (14). For that the origin of coordinates should be placed at the joint between the beams and put new variables along the axes of beams x_1 and x_2, which are connected with the initial (basic) coordinate system by the following ratio:

$$x_1 = x_k - x; \;\; x_2 = x - x_k . \quad (15)$$

For the semi-infinite beam with the lower stiffness C_{22} (Fig. 1), the equation (14) takes the form:

$$U_{22}(x_2) = U(0)\cdot B_1(x_2) + U'(0)\cdot B_2(x_2) + \\ + \frac{1}{EI_2}\left[-M(0)\cdot B_3(x_2) - Q(0)\cdot B_4(x_2) + \sum_i P_i^{22} B_4\left(x_2 - x_2^i\right)\right]. \quad (16)$$

For the second semi-infinite beam (rail) of the higher stiffness C_{21} (Fig. 1), the equation (14) takes the form:

$$U_{21}(x_2) = U(0)\cdot B_1(x_1) + U'(0)\cdot B_2(x_1) + \\ + \frac{1}{EI_1}\left[-M(0)\cdot B_3(x_1) - Q(0)\cdot B_4(x_2) + \sum_i P_i^{21} B_4\left(x - x_1^i\right)\right]. \quad (17)$$

Arbitrary constants $U(0)$, $U'(0)$ for the semi-infinite beam are determined from the condition that is at large values of x, which they are derived; movements should be zero [15]. Where $x \to \infty$, $ch\alpha x \to sh\alpha x \to \frac{1}{2}e^{\alpha x}$.

Using the equation (10) after the transformation we obtain the following:

$$\begin{cases} B_3(x) \to \frac{1}{2} e^{\alpha x}(\cos\alpha x), \\ B_2(x) \to \frac{1}{4\alpha} e^{\alpha x}(\sin\alpha x + \cos\alpha x), \\ B_3(x) \to \frac{1}{4\alpha^2} e^{\alpha x}(\sin\alpha x), \\ B_4(x) \to \frac{1}{8\alpha^3} e^{\alpha x}(\sin\alpha x - \cos\alpha x), \\ and\ if\ x = \left(x - x^i\right), \\ B_4(x)\left(x - x^i\right) \to \frac{1}{8\alpha^3} e^{\alpha\left(x - x^i\right)}\left[\sin\alpha\left(x - x^i\right) - \cos\alpha\left(x - x^i\right)\right]. \end{cases} \quad (18)$$

Using (18) the equation of the deflection of the beam (14) we reduce to the form:

$$U(x \to \infty) = \frac{1}{2} e^{\alpha x}\left\{\left\{\cos^{\alpha x}\cdot\left\{U(0) + U'(0)\frac{1}{2\alpha} + \frac{1}{EI}\times \right.\right.\right. \\ \left.\left.\times\left[\frac{Q(0)}{4\alpha^3} - \sum_i \frac{P_i e^{-\alpha x^i}}{4\alpha^3}\left(\sin\alpha x^i + \cos\alpha x^i\right)\right]\right\} + \sin\alpha x \times \\ \left.\left.\times\left\{U'(0)\frac{1}{2\alpha} + \frac{1}{EI}\cdot\left[\begin{matrix} -\frac{Q(0)}{4\alpha^3} - \frac{M(0)}{2\alpha^2}\times \\ \times\sum_i \frac{P_i e^{-\alpha x^i}}{4\alpha^3}\left(\cos\alpha x^i + \sin\alpha x^i\right)\end{matrix}\right]\right\}\right\}\right\} \to 0. \quad (19)$$

If the coefficients of the functions $\sin\alpha x$ and $\cos\alpha x$ at (19) are zero, displacements $U(x \to \infty)$ become zero.

To ensure this condition from the equation (19) it is necessary to allocate and make zero of two following equations:

$$\begin{cases} U(0) + U'(0)\frac{1}{2\alpha} + \frac{1}{EI}\times \\ \times\left[\frac{Q(0)}{4\alpha^3} - \sum_i \frac{P_i e^{-\alpha x^i}}{4\alpha^3}\left(\sin\alpha x^i + \cos\alpha x^i\right)\right] = 0, \\ U'(0) + U'(0)\frac{1}{2\alpha} + \frac{1}{EI}\times \\ \times\left[-\frac{Q(0)}{4\alpha^3} - \frac{M(0)}{2\alpha^2}\sum_i \frac{P_i e^{-\alpha x^i}}{4\alpha^3}\left(\cos\alpha x^i + \sin\alpha x^i\right)\right] = 0. \end{cases} \quad (20)$$

By solving the system of equations (20), we obtain the expression for determining the unknown initial parameters $U(0)$ and $U'(0)$ in the general form:

$$\begin{cases} U(0) = \frac{1}{EI}\cdot\left[\frac{Q(0)}{2\alpha^3} - \frac{M(0)}{2\alpha^2} + \sum_i \frac{P_i e^{-\alpha x^i}}{2\alpha^3}\cos\alpha x^i\right], \\ U'(0) = \frac{1}{EI}\cdot\left[-\frac{Q(0)}{2\alpha^2} - \frac{M(0)}{\alpha} - \sum_i \frac{P_i}{2\alpha^2}\left(\cos\alpha x^i + \sin\alpha x^i\right)\right]. \end{cases} \quad (21)$$

Using the equation (21) and the data of the scheme (Fig. 1) for the beam with a lower linear stiffness C_{22} we obtain:

$$\begin{cases} U_{22}(0)=\dfrac{1}{EI_2}\cdot\left[\begin{array}{l}-\dfrac{Q(0)}{2\alpha_{22}^3}-\dfrac{M(0)}{2\alpha_{22}^2}+\\ +\sum\limits_i\dfrac{P_i^{22}e^{-\alpha_{22}x_2^i}}{2\alpha_{22}^3}\cos\alpha_{22}x_2^i\end{array}\right], \\ U'_{22}(0)=\dfrac{1}{EI_2}\cdot\left[\begin{array}{l}\dfrac{Q(0)}{2\alpha_{22}^2}+\dfrac{M(0)}{2\alpha_{22}}-\\ -\sum\limits_i\dfrac{P_i^{22}e^{-\alpha_{22}x_2^i}}{2\alpha_{22}^2}\left(\cos\alpha_{22}x_2^i-\sin\alpha_{22}x_2^i\right)\end{array}\right]. \end{cases} \quad (22)$$

For beams with a higher linear stiffness C_{21} the equation will be of the form:

$$\begin{cases} U_{21}(0)=\dfrac{1}{EI_1}\cdot\left[\begin{array}{l}-\dfrac{Q(0)}{2\alpha_{21}^3}-\dfrac{M(0)}{2\alpha_{21}^2}+\\ +\sum\limits_i\dfrac{P_i^{21}e^{-\alpha_{21}x_1^i}}{2\alpha_{21}^3}\cos\alpha_{21}x_1^i\end{array}\right], \\ -U'_{21}(0)=\dfrac{1}{EI_1}\cdot\left[\begin{array}{l}\dfrac{Q(0)}{2\alpha_{21}^2}+\dfrac{M(0)}{2\alpha_{21}}-\\ -\sum\limits_i\dfrac{P_i^{21}e^{-\alpha_{21}x_1^i}}{2\alpha_{21}^2}\left(\cos\alpha_{21}x_1^i-\sin\alpha_{22}x_1^i\right)\end{array}\right]. \end{cases} \quad (23)$$

After making some changes, we obtain:

$$\left.\begin{array}{l} \varphi_{21}^i=\dfrac{1}{\alpha_{21}^3}e^{-\alpha_{21}x_1^i}\cos\alpha_{21}x_1^i, \\ q_{21}^i=\dfrac{1}{\alpha_{21}^2}e^{-\alpha_{21}x_1^i}\left(\cos\alpha_{21}x_1^i-\sin\alpha_{21}x_1^i\right), \\ \varphi_{22}^i=\dfrac{1}{\alpha_{22}^3}e^{-\alpha_{22}x_2^i}\cos\alpha_{22}x_2^i, \\ q_{22}^i=\dfrac{1}{\alpha_{22}^2}e^{-\alpha_{22}x_2^i}\left(\cos\alpha_{22}x_2^i-\sin\alpha_{22}x_2^i\right). \end{array}\right\} \quad (24)$$

We can simplify the expressions (22 and 23):

$$\begin{cases} U_{22}(0)=\dfrac{1}{EI_2}\cdot\left[-\dfrac{Q(0)}{2\alpha_{22}^3}-\dfrac{M(0)}{2\alpha_{22}^2}-\sum\limits_i\dfrac{1}{2}P_i^{22}\cdot\varphi_{22}^i\right], \\ U'_{22}(0)=\dfrac{1}{EI_2}\cdot\left[\dfrac{Q(0)}{2\alpha_{22}^2}+\dfrac{M(0)}{2\alpha_{22}}+\sum\limits_i\dfrac{1}{2}P_i^{22}\cdot\varphi_{22}^i\right]. \end{cases} \quad (25)$$

$$\begin{cases} U_{21}(0)=\dfrac{1}{EI_1}\cdot\left[-\dfrac{Q(0)}{2\alpha_{21}^3}-\dfrac{M(0)}{2\alpha_{21}^2}-\sum\limits_i\dfrac{1}{2}P_i^{21}\cdot\varphi_{21}^i\right], \\ -U'_{21}(0)=\dfrac{1}{EI_1}\cdot\left[\dfrac{Q(0)}{2\alpha_{21}^2}+\dfrac{M(0)}{2\alpha_{21}}+\sum\limits_i\dfrac{1}{2}P_i^{21}\cdot\varphi_{21}^i\right]. \end{cases} \quad (26)$$

Here $U_{21}(0)$, $U'_{21}(0)$, $U_{22}(0)$, $U''_{22}(0)$ – displacements and rotation angles as derivatives of the from U_{ij} of both parts of the beams (Fig. 1) at their joints.

Proceeding from the conditions of the equality of displacements $U(0)$ and rotation angles $U'(0)$, we can determine the initial parameters $Q(0)$ and $M(0)$ in (22, 23) at the joint of two semi-infinite beams.

From the conditions of compatibility in the adopted coordinate systems:

$$\begin{cases} U_{21}(0)=U_{22}(0), \\ -U'_{21}(0)=U'_{22}(0). \end{cases} \quad (27)$$

After changing values $U_{21}(0)$, $U_{22}(0)$, and also $U'_{21}(0)$, $U'_{22}(0)$ for expressions from (25, 26) and substituting in (27) we obtain:

$$\begin{cases} Q(0)\left(\dfrac{1}{\alpha_{21}^3}+\dfrac{1}{\alpha_{22}^3}\right)-M(0)\left(\dfrac{1}{\alpha_{21}^2}+\dfrac{1}{\alpha_{22}^2}\right)= \\ =\sum\limits_{i=0}^{S}P_i^{21}\cdot\varphi_{21}^i-\sum\limits_{i=S+x}^{n+1}P_i^{22}\cdot\varphi_{22}^i, \\ Q(0)\left(\dfrac{1}{\alpha_{21}^2}+\dfrac{1}{\alpha_{22}^2}\right)-M(0)\left(\dfrac{1}{\alpha_{21}}+\dfrac{1}{\alpha_{22}}\right)= \\ =\sum\limits_{i=0}^{S}P_i^{21}\cdot q_{21}^i-\sum\limits_{i=S+x}^{n+1}P_i^{22}\cdot q_{22}^i. \end{cases} \quad (28)$$

We made an additional replacement:

$$\left(\frac{1}{\alpha_{21}^3}+\frac{1}{\alpha_{22}^3}\right)=G_{11},$$

$$\left(\frac{1}{\alpha_{21}^2}+\frac{1}{\alpha_{22}^2}\right)=G_{12},\ \left(\frac{1}{\alpha_{21}}+\frac{1}{\alpha_{22}}\right)=G_{22},$$

$$G_{11}\cdot G_{22}-G_{12}^2=\Delta, \quad (29)$$

and solved the combined equations (28), taking into account the substitutions (29) relative to $Q(0)$ and $M(0)$ we get initial parameters which are explicitly expressed by concentrated forces P_i^{21} and P_i^{22}:

$$\begin{cases} Q(0) = \dfrac{1}{\Delta} \cdot \left[\sum\limits_{i=0}^{S} P_i^{21} \left(G_{22}\varphi_{21}^{i} - G_{12} g_{21}^{i} \right) - \right. \\ \left. - \sum\limits_{i=S+x}^{n+1} P_i^{22} \left(G_{22}\varphi_{22}^{i} - G_{12} g_{22}^{i} \right) \right], \\ M(0) = \dfrac{1}{\Delta} \cdot \left[\sum\limits_{i=0}^{S} P_i^{21} \left(G_{12}\varphi_{21}^{i} - G_{11} g_{21}^{i} \right) - \right. \\ \left. - \sum\limits_{i=S+x}^{n+1} P_i^{22} \left(G_{12}\varphi_{22}^{i} - G_{11} g_{22}^{i} \right) \right]. \end{cases} \tag{30}$$

Substituting the initial parameters $Q(0)$ and $M(0)$ from (30) to (25 and 26) and performing some transformations we obtain:

$$\begin{aligned} U_{22}(0) = \frac{1}{2EI_2\Delta} \cdot \Bigg\{ \sum P_i^{21} \left[-\varphi'^{i}_{21} \left(\frac{G_{22}}{\alpha_{22}^3} + \frac{G_{12}}{\alpha_{22}^3} \right) + q'^{i}_{22} \left(\frac{G_{12}}{\alpha_{22}^3} + \frac{G_{11}}{\alpha_{22}^3} \right) \right] + \\ + \sum P_i^{22} \left[\varphi'^{i}_{22} \left(\frac{G_{22}}{\alpha_{22}^3} + \frac{G_{12}}{\alpha_{22}^3} - \Delta \right) + q'^{i}_{21} \left(\frac{G_{12}}{\alpha_{22}^3} + \frac{G_{11}}{\alpha_{22}^3} \right) \right] \Bigg\}, \\ U'_{22}(0) = \frac{1}{2EI_2\Delta} \Bigg\{ \sum P_i^{21} \left[\varphi'^{i}_{21} \left(\frac{G_{22}}{\alpha_{22}^2} + \frac{2G_{12}}{\alpha_{22}} \right) - q'^{i}_{21} \left(\frac{G_{12}}{\alpha_{22}^2} + \frac{2G_{11}}{\alpha_{22}} \right) \right] + \\ \sum P_i^{22} \left[-\varphi'^{i}_{22} \left(\frac{G_{22}}{\alpha_{22}^2} + \frac{2G_{12}}{\alpha_{22}} - \Delta \right) - q'^{i}_{22} \left(\frac{G_{12}}{\alpha_{22}^2} + \frac{2G_{11}}{\alpha_{22}} - \Delta \right) \right] \Bigg\}. \end{aligned} \tag{31}$$

$$\begin{aligned} U_{21}(0) = \frac{1}{2EI_1\Delta} \cdot \Bigg\{ \sum P_i^{21} \left[\varphi'^{i}_{21} \left(\frac{G_{22}}{\alpha_{21}^3} - \frac{G_{12}}{\alpha_{21}^2} - \Delta \right) - q'^{i}_{21} \left(\frac{G_{12}}{\alpha_{21}^3} + \frac{G_{11}}{\alpha_{22}^3} \right) \right] - \\ - \sum P_i^{22} \left[\varphi'^{i}_{22} \left(\frac{G_{22}}{\alpha_{21}^3} - \frac{G_{12}}{\alpha_{21}^2} \right) + q'^{i}_{22} \left(\frac{G_{12}}{\alpha_{21}^3} + \frac{G_{11}}{\alpha_{21}^3} \right) \right] \Bigg\}, \end{aligned} \tag{32}$$

$$\begin{aligned} U'_{21}(0) = \frac{1}{2EI_1\Delta} \Bigg\{ \sum P_i^{21} \left[-\varphi'^{i}_{21} \left(\frac{G_{22}}{\alpha_{21}^2} - \frac{2G_{12}}{\alpha_{21}} \right) + q'^{i}_{21} \left(\frac{G_{12}}{\alpha_{22}^2} - \frac{2G_{11}}{\alpha_{21}} + \Delta \right) \right] + \\ \sum P_i^{22} \left[\varphi'^{i}_{22} \left(\frac{G_{22}}{\alpha_{21}^2} - \frac{2G_{12}}{\alpha_{21}} \right) - q'^{i}_{22} \left(\frac{G_{12}}{\alpha_{21}^2} - \frac{2G_{11}}{\alpha_{21}} \right) \right] \Bigg\}. \end{aligned}$$

For simplicity, we are making the replacement:

$$\begin{cases} U_{21,1}^{i} = \dfrac{1}{2\Delta} \left[\varphi'^{i}_{21} \left(\dfrac{G_{22}}{\alpha_{21}^3} - \dfrac{G_{12}}{\alpha_{21}^2} - \Delta \right) - q'^{i}_{21} \left(\dfrac{G_{12}}{\alpha_{21}^3} + \dfrac{G_{11}}{\alpha_{21}^3} \right) \right], \\ U'_{21,2} = \dfrac{1}{2\Delta} \left[-\varphi'^{i}_{21} \left(\dfrac{G_{22}}{\alpha_{21}^3} - \dfrac{G_{12}}{\alpha_{21}^2} \right) - q'^{i}_{22} \left(\dfrac{G_{12}}{\alpha_{21}^3} + \dfrac{G_{11}}{\alpha_{21}^2} \right) \right]. \end{cases} \tag{33}$$

$$\begin{cases} U_{22,2}^{i} = \dfrac{1}{2\Delta} \left[-\varphi'^{i}_{21} \left(\dfrac{G_{22}}{\alpha_{22}^3} + \dfrac{G_{12}}{\alpha_{22}^2} \right) + q'^{i}_{21} \left(\dfrac{G_{12}}{\alpha_{22}^3} + \dfrac{G_{11}}{\alpha_{22}^3} \right) \right], \\ U'_{2,2,2} = \dfrac{1}{2\Delta} \left[-\varphi'^{i}_{22} \left(\dfrac{G_{22}}{\alpha_{22}^3} - \dfrac{G_{12}}{\alpha_{22}^2} - \Delta \right) + q'^{i}_{22} \left(\dfrac{G_{12}}{\alpha_{22}^3} + \dfrac{G_{11}}{\alpha_{22}^2} \right) \right]. \end{cases} \tag{34}$$

$$\begin{cases} U_{21,1}^{i} = \dfrac{1}{2\Delta} \left[-\varphi'^{i}_{21} \left(\dfrac{G_{22}}{\alpha_{22}^2} + \dfrac{G_{12}}{\alpha_{22}} \right) + q'^{i}_{21} \left(\dfrac{G_{12}}{\alpha_{21}^2} - \dfrac{2G_{11}}{\alpha_{21}} + \Delta \right) \right], \\ U'_{21,2} = \dfrac{1}{2\Delta} \left[\varphi'^{i}_{22} \left(\dfrac{G_{22}}{\alpha_{21}^2} - \dfrac{G_{12}}{\alpha_{21}^2} \right) - q'^{i}_{22} \left(\dfrac{G_{12}}{\alpha_{21}^2} + \dfrac{2G_{11}}{\alpha_{21}} \right) \right]. \end{cases} \tag{35}$$

Also in the formulas (30):

$$Q'^{i}_{21} = \frac{1}{\Delta} \left(G_{22}\varphi'^{i}_{21} - G_{12}\varphi'^{i}_{21} \right); \quad {}^{i}_{22} = \frac{1}{\Delta} \left(-G_{22}\varphi'^{i}_{22} - G_{12}\varphi'^{i}_{22} \right), \tag{36}$$

$$M'^{i}_{21} = \frac{1}{\Delta} \left(G_{12}\varphi'^{i}_{21} - G_{11}\varphi'^{i}_{21} \right); \quad M'^{i}_{22} = \frac{1}{\Delta} \left(-G_{12}\varphi'^{i}_{22} - G_{11}\varphi'^{i}_{22} \right). \tag{37}$$

Substituting in (30) we obtained simplified expressions from (36, 37) designations for initial parameters $U(0)$, $U'(0)$, $Q(0)$, $M(0)$:

$$Q(0) = \sum_{i=0}^{S} P_i^{21} Q_{21}^{i} + \sum_{i=S+1}^{n+1} P_i^{22} Q_{22}^{i}, \tag{38}$$

$$M(0) = \sum_{i=0}^{S} P_i^{21} Q_{21}^{i} + \sum_{i=S+1}^{n+1} P_i^{22} M_{22}^{i}.$$

Similarly, transforming (25 and 26), using (33-37) we obtain:

$$\begin{cases} U_{21}(0) = \dfrac{1}{EI_1} \left[\sum\limits_{i=0}^{S} P_i^{21} U_{21,1}^{i} + \sum\limits_{i=S+1}^{n+1} P_i^{22} U_{21,2}^{i} \right], \\ U_{22}(0) = \dfrac{1}{EI_2} \left[\sum\limits_{i=0}^{S} P_i^{21} U_{22,1}^{i} + \sum\limits_{i=S+1}^{n+1} P_i^{22} U_{22,2}^{i} \right]. \end{cases} \tag{39}$$

$$\begin{cases} U'_{21}(0) = \dfrac{1}{EI_1} \left[\sum\limits_{i=0}^{S} P_i^{21} U'^{i}_{21,1} + \sum\limits_{i=S+1}^{n+1} P_i^{22} U'^{i}_{21,2} \right], \\ U'_{22}(0) = \dfrac{1}{EI_2} \left[\sum\limits_{i=0}^{S} P_i^{21} U'^{i}_{22,1} + \sum\limits_{i=S+1}^{n+1} P_i^{22} U'^{i}_{22,2} \right]. \end{cases} \tag{40}$$

Results of the transformation the equations (38-40) allow to represent all the initial parameters of two semi-infinite rigidly coupled of different stiffness beams laying on the elastic foundation via concentrated forces P_i^{21}, P_i^{22}, acting from the railway vehicles.

From the condition of the compatibility of displacements (27) the expression $U_{21}(0)=U_{22}(0)$, $-U'_{21}=U'_{22}$ is hold and used in further calculations.

Using the desired expression (16, 17) for the displacement of two semi-infinite beams which are rigidly connected and have different stiffness in the unfolded state, with regard to (38-40), we obtain:

$$
\begin{aligned}
U_{21}(x_1)=&\frac{1}{EI_1}\Bigg\{\sum_{i=0}^{S}P_i^{21}\Big[\Big(U_{21,1}^{i}\cdot B_1(x_1)-U_{21,1}^{\prime i}\cdot B_2(x_1)-\\
&-M_{21}^{\prime i}\cdot B_3(x_1)+Q_{21}^{i}\cdot B_4(x_1)\Big)\Big]\_|^{-}\\
&(x_1-0)-B_4\left(x_1-x_1^{i}\right)\_|^{-}(x_1-x_i)+\\
&+\sum_{i=S+1}^{n+1}P_i^{22}\Big[\Big(U_{21,2}^{i}\cdot B_1(x_1)-U_{21,1}^{\prime i}\cdot B_2(x_1)-\\
&-M_{22}^{\prime i}\cdot B_3(x_1)+Q_{22}^{\prime i}\cdot B_4(x_1)\Big)\Big]\_|^{-}(x_1-0)\Bigg\}.
\end{aligned}
\qquad (41)
$$

$$
\begin{aligned}
U_{22}(x_2)=&\frac{1}{EI_2}\Bigg\{\sum_{i=0}^{S}P_i^{21}\Big[\Big(U_{22,1}^{i}\cdot B_1(x_2)-U_{22,1}^{\prime i}\cdot B_2(x_2)-\\
&-M_{21}^{\prime i}\cdot B_3(x_2)+Q_{21}^{\prime i}\cdot B_4(x_2)\Big)\Big]\_|^{-}(x_2-0)+\\
&+\sum_{i=S+1}^{n+1}P_i^{22}\Big[\Big(\Big(U_{22,2}^{i}\cdot B_1(x_2)-U_{22,2}^{\prime i}\cdot B_2(x_2)\Big)-\\
&-M_{22}^{\prime i}\cdot B_3(x_2)+Q_{22}^{\prime i}\cdot B_4(x_2)\Big)\Big]\_|^{-}(x_2-0)-\\
&-B_4\left(x_2-x_2^{i}\right)\_|^{-}\left(x_2-x_2^{i}\right)\Bigg\}.
\end{aligned}
\qquad (42)
$$

When passing from the local to the global coordinate system in Heaviside record step function making changes using the expression $x_1=x_k-x;$ $x_2=x-x_k$ and regulations:

$$
\_|^{-}(x_k-x)=\begin{cases}1, at\ \tilde{o}\le\tilde{o}_k,\\ 0, at\ \tilde{o}\ge\tilde{o}_k;\end{cases}
\qquad
\_|^{-}\left(x^{i}-x\right)=\begin{cases}1, at\ \tilde{o}\le\tilde{o}^{i},\\ 0, at\ \tilde{o}\ge\tilde{o}^{i}.\end{cases}
\qquad (43)
$$

Making the transfer of the equation (41, 42) to the global coordinate system x we are taking to the account the expressions in [2]:

$$
\begin{aligned}
U_{21}(x_1)=&\frac{1}{EI_1}\Bigg\{\sum_{i=0}^{S}P_i^{21}\Big[\Big(U_{21,1}^{i}\cdot B_1(x_k-x)-U_{21,1}^{\prime i}\cdot B_2(x_k-x)-\\
&-M_{21}^{\prime i}\cdot B_3(x_k-x)+Q_{21}^{i}\cdot B_4(x_k-x)\Big)\Big]\_|^{-}(x_k-x)-\\
&-B_4\left(x^{i}-x\right)\_|^{-}\left(x^{i}-x\right)+\sum_{i=S+1}^{n+1}P_i^{22}\Big[\Big(U_{21,2}^{i}\cdot B_1\left(x^{i}-x\right)-\\
&-U_{21,1}^{\prime i}\cdot B_2\left(x^{i}-x\right)-M_{22}^{\prime i}\cdot B_3\left(x^{i}-x\right)+Q_{22}^{i}\cdot B_4\left(x^{i}-x\right)\Big)\Big]\_|^{-}\\
&\left(x^{i}-x\right)\Bigg\}.
\end{aligned}
\qquad (44)
$$

$$
\begin{aligned}
U_{22}(x_2)=&\frac{1}{EI_2}\Bigg\{\sum_{i=0}^{S}P_i^{21}\Big[\Big(U_{22,1}^{i}\cdot B_1(x-x_k)-\\
&-U_{22,1}^{\prime i}\cdot B_2(x-x_k)-M_{21}^{\prime i}\cdot B_3(x-x_k)+Q_{21}^{i}\cdot B_4(x-x_k)\Big)\Big]\_|^{-}\\
&(x-x_k)+\sum_{i=S+1}^{n+1}P_i^{22}\Big[\Big(U_{22,2}^{i}\cdot B_1(x-x_k)-U_{22,2}^{\prime i}\cdot B_2(x-x_k)-\\
&-M_{22}^{\prime i}\cdot B_3(x-x_k)+Q_{22}^{i}\cdot B_4(x-x_k)\Big)\Big]\_|^{-}(x-x_k)-\\
&-B_4\left(x-x^{i}\right)\_|^{-}\left(x-x^{i}\right)\Bigg\}.
\end{aligned}
\qquad (45)
$$

When transferring the force P along the rail (beam) of the variable stiffness laying on an elastic foundation (i.e., rolling the wheels of the rolling stock), from semi-infinite beam section with higher stiffness to the section with less stiffness, after the load jump the perturbation is damped with the increasing distance from the considered cross section at the point x_k to the infinity.

According to these equations (43) and (44) of the function $B_1(x)........B_4(x)$, taking into account that the fading should be of the following form:

$$
\begin{cases}
B_1(x)\to\dfrac{1}{2}e^{-\alpha x}(\cos\alpha x),\\
B_2(x)\to\dfrac{1}{4\alpha}e^{-\alpha x}(\sin\alpha x-\cos\alpha x),\\
B_3(x)\to\dfrac{1}{4\alpha^2}e^{-\alpha x}(\sin\alpha x),\\
B_4(x)\to\dfrac{1}{8\alpha^3}e^{-\alpha x}(\sin\alpha x+\cos\alpha x)\ \text{or}\\
B_4(x-x_k)\to\dfrac{1}{8\alpha^3}e^{-\alpha(x-x_k)}\left[\sin\alpha(x-x_k)+\cos\alpha x-x_k\right].
\end{cases}
\qquad (46)
$$

Using Krylov’s functions we obtain the equation of the deformation parameters of the beam with the damped oscillations, taking into account [9]:

$$U(x) = U(0)ch\alpha x \cdot \cos\alpha x + \\ + U'(0)\frac{1}{2\alpha}[ch\alpha x \cdot \sin\alpha x + sh\alpha x \cdot \cos\alpha x] + \\ + U''(0)\frac{1}{2\alpha^2}sh\alpha x \cdot \sin\alpha x + \\ + U'''(0)\frac{1}{4\alpha^3}[ch\alpha x \cdot \sin\alpha x - sh\alpha x \cdot \cos\alpha x] + \\ + \frac{P_i}{EI_2}\frac{1}{4\alpha^3}[ch\alpha(x - x_i) \cdot \sin\alpha(x - x_i) - \\ - sh\alpha(x - x_i) \cdot \cos\alpha(x - x_i)]. \quad (47)$$

And taking into account that $ch\alpha x = \frac{1}{2}\left(e^{\alpha x} + e^{-\alpha x}\right)$; $sh\alpha x = \frac{1}{2}\left(e^{\alpha x} - e^{-\alpha x}\right)$ it can be presented to the form with the lower stiffness for the semi-infinite beam:

$$U(x) = \frac{1}{2}e^{\alpha x}\Big\{U(0)\cos\alpha x + U'(0)\frac{1}{2\alpha}(\sin\alpha x + \cos\alpha x) + \\ + U''(0)\frac{1}{2\alpha^2}\sin\alpha x + U'''(0)\frac{1}{4\alpha^3}(\sin\alpha x - \cos\alpha x) + \\ + \frac{P_i}{EI_2}\frac{1}{4\alpha^3}\Big[e^{-\alpha x}(\sin\alpha x \cdot \cos\alpha x_i - \cos\alpha x \cdot \sin\alpha x_i) - \\ - e^{-\alpha x_i}(\cos\alpha x \cdot \sin\alpha x_i + \sin\alpha x \cdot \cos\alpha x_i)\Big]\Big\} + \\ \frac{1}{2}e^{\alpha x}\{U(0)\cos\alpha x + U'(0)\frac{1}{2\alpha}(\sin\alpha x - \cos\alpha x) + \\ + U''(0)\frac{1}{2\alpha^2}\sin\alpha x + U'''(0)\frac{1}{4\alpha^3}(\sin\alpha x + \cos\alpha x) + \\ + \frac{P_i}{EI_2}\frac{1}{4\alpha^3}\Big[e^{-\alpha x_i}(\sin\alpha x \cdot \cos\alpha x_i - \cos\alpha x \cdot \sin\alpha x_i) + \\ + e^{-\alpha x_i}(\cos\alpha x \cdot \cos\alpha x_i + \cos\alpha x \cdot \cos\alpha x_i)\Big]\Big\}. \quad (48)$$

With large values of x, while damping of amplitudes of displacements, expressions in braces (with multiplier $e^{\alpha x}$) can be ignored because they are under common factors $\sin\alpha x$ and $\cos\alpha x$. This condition is taken into account while obtaining the expression for $U(0)$ and $U'(0)$.

The expression containing the factor $e^{\alpha x}$, defines functions $B_1(x)....B_4(x)$ and $B_4(x - x_i)$ factors with the initial parameters $U(0)...U'''(0)$ and $\frac{P_i}{EI_{1,2}}$.

Thus equations (44, 45) allow to define displacements of the beam of the variable stiffness, laying on the elastic foundation, which varies accordingly to the stepped law (Fig. 2). Also the data of the equation for a semi-infinite beams ensure that the conditions of compatibility of displacements (linear and angular) at the point of junction, where the abrupt change of their stiffness happens; equations (47, 48) obtained after transformation of the explicitly contain only external concentrated forces P_i that act from the side of railway vehicles.

The above expressions used for the displacing of the semi-infinite beam laying on the elastic foundation are valid for arbitrary values of stiffness C and the elastic foundation.

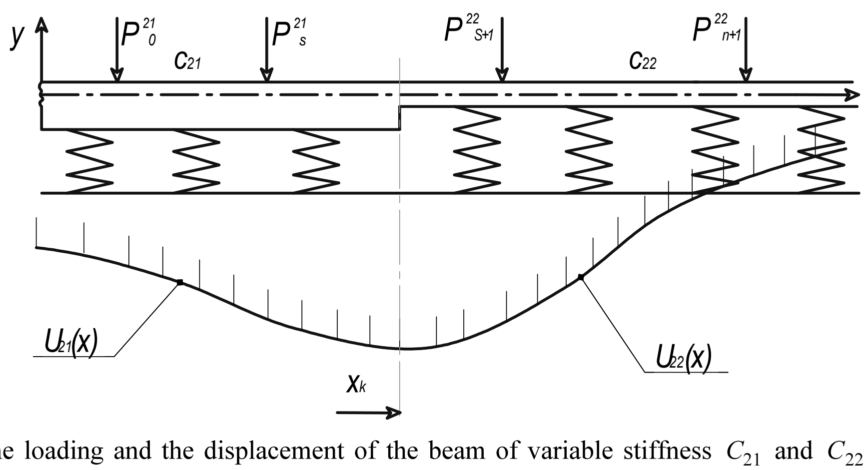

Fig. 2. Scheme of the loading and the displacement of the beam of variable stiffness C_{21} and C_{22} on the elastic foundation

In case, if the magnitude of the beam (rail) Δc is relatively small, it is convenient to use the method of the small parameter [6 ,8, 11, 21], which allows to reduce the number of computations in determining the displacements of the beam on the elastic foundation. The differential equation describing the stress-strain state of the beam with the lower stiffness while it's spasmodic change, laying on the elastic foundation is:

$$EI_2\frac{d^4U^2}{dx^4}+c_{22}\left[1+\varepsilon\underline{|}^{-}(x-x_k)\right]\cdot U^2(x)=-\sum P_i\delta(x-x_i), \quad (49)$$

where: $\varepsilon=\dfrac{c_{21}-c_{22}}{c_{22}}$.

Series expansion $U^2(x)$ by the small parameter ε [6]

$$U^2(x)=U_0^2(x)+\varepsilon U_1^2(x)+\varepsilon^2U_2^2(x)........, \quad (50)$$

By substituting the solution (50) into the equation (49) we obtain:

$$EI_2\frac{d^4U^2}{dx^4}\left[U^2(x)=U_0^2(x)+\varepsilon U_1^2(x)+\varepsilon^2U_2^2(x)+......\right]+$$
$$+c_2\left[1+\varepsilon\underline{|}^{-}(x-x_k)\right]\left[U^2(x)=U_0^2(x)+\varepsilon U_1^2(x)+\right.$$
$$\left.+\varepsilon^2U_2^2(x)+.....\right]=\sum P_i\delta(x-x_k). \quad (51)$$

By grouping the terms of the equation (51) by the degree ε^j, j we obtain:

$$\varepsilon^0\left[EI_2\frac{d^4U_0^2}{dx^4}c_2U_0^2(x)+\sum_{i=0}^{n+1}P_i\delta(x-x_k)\right]+$$
$$+\varepsilon^1\left[EI_2\frac{d^4U_1^2}{dx^4}+c_2U_1^2(x)+c_2\underline{|}^{-}(x-x_k)\right]+$$
$$+\varepsilon^2\left[EI_2\frac{d^4U_2^2}{dx^4}c_2U_2^2(x)+c_2\underline{|}^{-}(x-x_k)U_1^2(x)\right]+......=0, \quad (52)$$

The expression with ε^j of different forces becomes zero. As a result, we obtain a system of differential equations:

$$\left.\begin{aligned}
&\frac{d^4U_0^2}{dx^4}+4\alpha^4U_0^2(x)=-\frac{1}{EI_2}\sum P_i\delta(x-x_k)=0,\\
&\frac{d^4U_1^2}{dx^4}+4\alpha^4U_1^2(x)=-4\alpha^4\underline{|}^{-}(x-x_k)U_0^2(x),\\
&\frac{d^4U_2^2}{dx^4}+4\alpha^4U_2^2(x)=-4\alpha^4\underline{|}^{-}(x-x_k)U_1^2(x),
\end{aligned}\right\} \quad (53)$$

where: α^4 for the beam side with the lower stiffness:

$$\alpha^4=\frac{1}{4}\frac{c_2}{EI_2}. \quad (54)$$

The system of differential equations (53) is solving sequentially, beginning with the first equation of the system. The solution of the previous equation is included into the right side of the following equation, etc. [20].

Using expressions (44, 45) or (50) it is possible to sum the interaction of railway vehicles with the track structure, completed with rails of different brands (i.e. different linear stiffness), laying on the elastic foundation [4, 22].

Besides the classical approach of identifying the technical parameters of the deformation of the track panels laying on the elastic foundation with low values of the abrupt change of the stiffness of the rail the small parameter method can be used.

The purpose of this calculation (such an approach) is confirmed (checking) by the approach to the task solution of Winkler model as quiet simple and in many cases provides a good convergence with the practice.

It should be noted that the accuracy of determining the parameters of the permanent way, the method of calculations that was given above, essentially depends on the type of ties, which are the intermediate supports between the rail and under-rail space (ballast layer).

According to this while calculating it is necessary to make the definite assumption. For example, assuming that the load on the rail from the wheel of the rolling stock to the load of the transmitted to the resilient space, it means that the displacement of the boundary of the half-space can be determined by the formula:

$$U_3(x_1, x_2, 0) = \frac{1-\nu}{2\pi G_n} \frac{p^j}{r}, \qquad (55)$$

where: ν – Poisson's ratio, G_n – the modulus of stiffness of the half space, p^j – the load on the elastic half space from the track.

$$r^2 = x_1^2 + x_2^2, \qquad (56)$$

where: $r(x_1, x_2)$ – the distance between the point of the application of the force p^j and the point of determining the displacement.

To improve the accuracy of the task solution, we take into account the interaction of the track support with the under-rail elastic half-space boundary. In this case it is necessary to move to a distributed load $g_3(\xi_1, \xi_2)$. The displacement of the border the border of the elastic half-space can be defined by the following dependency:

$$U_3(x_1, x_2, 0) = \frac{1-\nu}{2\pi G_n} \int_{\Sigma} \int \frac{g_3^i(\xi_1, \xi_2) d\xi_1, d\xi_2}{r_1}, \qquad (57)$$

where: ξ_1, ξ_2 – coordinates of different points on the side with the distributed load $g_3^j(\xi_1, \xi_2)$, $r_1^2 = (x_1 - \xi_1)^2 + (x_2 - \xi_2)^2$, at that r_1 – (variable), the distance between different points (ξ_1, ξ_2) of the loaded side to the point, which displacement is determined by $(x_1, x_2, x_3 = 0)$.

Formulas from the given above techniques allow to describe (but in practice) simple types of rail connections with three-dimensional space under-rail system. Despite bearing the rail on the elastic foundation and various analytical forms of description, the general scheme of calculation remains unchanged and allows determining the displacement of the rail track from work of concentrated forces p_3^j from the wheels of the vehicle. That is the part of an overall dual problem and requires taking into account the vertical displacement of the wheels of the railway vehicle and then maybe the solution of the dual task: railway vehicles, the track and the ballast.

CONCLUSIONS

1. We developed the generalized model of the adjoin formulation for the track and the elastic foundation with the load from the wheels of the railway vehicles, which allows to determine the stress-strain state in the component parts of a complex system.

2. The generalized model can be used in various designs of railway vehicles and different types of elastic foundation (simulated by Winkler foundation). We found the system of algebraic equations composed for the unknown forces of the wheel-rail interaction.

3. The result of the solution of the system of the linear equations interaction forces of the wheel-rail allow to make calculations of stresses, strains and displacements of all system components.

4. We developed a simplified model that allows (while operating on the rail system canvas of concentrated forces, on the rails) to determine the stress, the strain and the displacement of the rail (simulated infinite beam) on Winkler foundation with varying stiffness. It has two solutions. The first is based on the "conjugation" of two semi-infinite beams (rails) with a step change of stiffness on the elastic foundation. Such a solution of the task is "exact" and taking into account different stiffness values of connected rails. The basis of the second solution is the method of the "small" parameter. The given approach is effective when the stiffness characteristics of two adjacent sections slightly differ from one another.

5. To estimate the allowable using of generalized Winkler foundation we obtained analytical expressions of the displacements for the same options which were obtained for the elastic half-space.

6. The proposed generalized model of Winkler foundation allows effective solving of many important practical problems, including the problem of the deformation of the rail with the elastic half-space and choose its stiffness characteristics to reduce the vertical dynamic

component of force generated while transition of the rail wheel with one stiffness on the rail with the other stiffness.

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МАТЕМАТИЧЕСКАЯ МОДЕЛЬ ДЕФОРМАЦИИ РЕЛЬСОШПАЛЬНОЙ РЕШЕТКИ СО СТУПЕНЧАТЫМ ИЗМЕНЕНИЕМ ЖЕСТКОСТИ НА УПРУГОМ ВИНКЛЕРОВОМ ОСНОВАНИИ ПОСТОЯННОЙ ЖЕСТКОСТИ

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А н н о т а ц и я . Рассмотрена постановка задачи построения математической модели деформации рельсошпальной решетки со ступенчатым изменением жесткости на упругом винклеровом основании. Рельсовая нить представлена в виде балки переменного сечения. Для определения перемещений и углов поворота сечений в месте стыковки использован метод преобразования уравнения Лапласа и функция Дирака. В результате моделирования получены выражения для перемещения полубесконечных балок (состыкованных рельсов), лежащих на упругом основании, которые могут быть использованы для произвольных значений жесткости «С» упругого основания. Предложен так же упрощенный вариант решения задачи с использованием метода «Малого» параметра для случая, когда жесткостные характеристики двух смежных участков рельса отличаются незначительно.

К л ю ч е в ы е с л о в а : подвижной состав, рельс, линейная жесткость, прогиб и угол поворота сечения в месте стыка, жесткость подрельсового основания, винклерово основание.