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M–BETTER BLOCK DESIGN IN BINARY CLASS $D(v=4w, b=6, n=3v)$

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Summary

In this paper we consider a certain class of block designs used in situations of limited experimental resources. We present properties of some binary partially balanced block designs. We prove that the proposed design is M–better than others in the binary subclass $D(v=4w, b=6, n=3v)$. This result makes it possible to formulate the construction of binary block designs as an optimization problem, and is a response to questions from agricultural researchers.

Key words and phrases: block design, M–better design, group divisible block design

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1. Introduction

Agricultural experiments are frequently field experiments involving one–way elimination of heterogeneity. The structure and scale of the experimental material are an important aspect of the planning of the experiment. Many such experiments have to be limited in terms of the numbers of blocks and of experimental units in each. In this situation, incomplete block designs with at most six blocks are used. These are designs with a binary incidence matrix, mostly with blocks of equal sizes and equireplicated treatments.

In this paper we describe a special construction problem for the arrangement of treatments on experimental units, in response to questions from agricultural researchers.

2. Background

In this section we study binary block designs where there are six blocks. We consider only connected block designs, with the meaning that a chain in such a design is a sequence of experimental units such that two consecutive units share either the same treatment or the same block, but not both.

We have v treatments arranged on n=3v experimental units which are grouped in b=6 blocks (B_1, …, B_6) of the same size in the following way:

B_1:	1	2	…	w	w+1	w+2	…	2w
B_2:	1	2	…	w	2w+1	2w+2	…	3w
B_3:	1	2	…	w	3w+1	3w+2	…	4w
B_4:	w+1	w+2	…	2w	2w+1	2w+2	…	3w
B_5:	w+1	w+2	…	2w	3w+1	3w+2	…	4w
B_6:	2w+1	2w+2	…	3w	3w+1	3w+2	…	4w

where 1,2, …,w, …,2w, …,3w, …,4w are numbers of treatments (w>1, v=4w, n=3v=12w). The above scheme of arrangement of treatments on experimental units defines a binary connected block design $d^*\in D(v=4w,\ b=6,\ n=3v)$. It is easy to see that for each pair of blocks w treatments are the same; w is the intersection number for all blocks. The block design is based on v=4w treatments (w>1) being arranged into 4 groups of w treatments each. Each treatment occurs in r=3 blocks, and two treatments belonging to the same w–dimensional group are such that they occur together in $\lambda_1=3$ blocks and are called first associates. When two treatments belong to different groups they occur together in $\lambda_2=1$ block and are called second associates. The block design d^* for w>1 is a group divisible block design. (It is known that when w=1 then $\lambda_1=\lambda_2=\lambda$ and we have a balanced incomplete block design.) The group divisible block design was defined by Bose and Conor (1952). Statistical properties of this design have been discussed by many authors, for example in recent years by Brzeskwiniewicz and Krzyszkowska (2009), Bagchi and Bagchi (2001), Das (2002) and Bagchi (2004). The design d^* has been described, for example, in theorem 7.3.1 of Caliński and Kageyama

(2003) and earlier in theorem 8.5.1 of Raghavarao (1971). The information matrix of design d^* has the following form:

$$\mathbf{C}_{d*} = 3\mathbf{I}_v - \frac{1}{2w}\mathbf{NN}' =$$

$$= \begin{pmatrix} 3\mathbf{I}_w - \frac{3}{2w}\mathbf{1}_w\mathbf{1}_w' & -\frac{1}{2w}\mathbf{1}_w\mathbf{1}_w' & -\frac{1}{2w}\mathbf{1}_w\mathbf{1}_w' & -\frac{1}{2w}\mathbf{1}_w\mathbf{1}_w' \\ -\frac{1}{2w}\mathbf{1}_w\mathbf{1}_w' & 3\mathbf{I}_w - \frac{3}{2w}\mathbf{1}_w\mathbf{1}_w' & -\frac{1}{2w}\mathbf{1}_w\mathbf{1}_w' & -\frac{1}{2w}\mathbf{1}_w\mathbf{1}_w' \\ -\frac{1}{2w}\mathbf{1}_w\mathbf{1}_w' & -\frac{1}{2w}\mathbf{1}_w\mathbf{1}_w' & 3\mathbf{I}_w - \frac{3}{2w}\mathbf{1}_w\mathbf{1}_w' & -\frac{1}{2w}\mathbf{1}_w\mathbf{1}_w' \\ -\frac{1}{2w}\mathbf{1}_w\mathbf{1}_w' & -\frac{1}{2w}\mathbf{1}_w\mathbf{1}_w' & -\frac{1}{2w}\mathbf{1}_w\mathbf{1}_w' & 3\mathbf{I}_w - \frac{3}{2w}\mathbf{1}_w\mathbf{1}_w' \end{pmatrix}, \tag{2.1}$$

where $\mathbf{N}$ is the incidence matrix, $\mathbf{I}_x$ is the unit matrix of order x and $\mathbf{1}_x$ is the x–dimensional vector of ones. The matrix $\mathbf{NN}'$ is the associate matrix of the form

$$\mathbf{NN}' = r\mathbf{A}_0 + \lambda_1\mathbf{A}_1 + \lambda_2\mathbf{A}_2,$$

where $\mathbf{A}_0 = \mathbf{I}_v$, $\mathbf{A}_1 = \mathbf{I}_4 \otimes (\mathbf{1}_w\mathbf{1}_w' - \mathbf{I}_w)$, $\mathbf{A}_2 = (\mathbf{1}_4\mathbf{1}_4' - \mathbf{I}_4) \otimes \mathbf{1}_w\mathbf{1}_w'$ and $\otimes$ denotes the Kronecker product of matrices, λ_1=3 and λ_2=1. The eigenvalues of the information matrix $\mathbf{C}_{d*}$ are γ_0=0, $\gamma_{1*} = 2$ and $\gamma_{2*} = 3$ for any v and w>1, with multiplicities 1, 3 and (v–4), respectively. So the v–1 dimensional vector of non–zero eigenvalues of matrix (2.1) is denoted by $\boldsymbol{\gamma}_{d*} = [\gamma_{1*}, \gamma_{1*}, \gamma_{1*}, \gamma_{2*}, ..., \gamma_{2*}]$, and the smallest non–zero eigenvalue for d^*, denoted as γ_{1*}, is equal to 2.

Let us now consider v treatments arranged on n=3v experimental units which are grouped in b=6 blocks (B_1, …, B_6), but in a different manner than above :

B_1:	1	2	…	w	$2w+1$	$2w+2$	…	$3w$
B_2:	1	2	…	w	$2w+1$	$2w+2$	…	$3w$
B_3:	1	2	…	w	$3w+1$	$3w+2$	…	$4w$
B_4:	$w+1$	$w+2$	…	$2w$	$2w+1$	$2w+2$	…	$3w$
B_5:	$w+1$	$w+2$	…	$2w$	$3w+1$	$3w+2$	…	$4w$
B_6:	$w+1$	$w+2$	…	$2w$	$3w+1$	$3w+2$	…	$4w$.

This scheme of arrangement of treatments on experimental units defines a binary connected block design $\mathrm{d}^x \in D(v=4w,\ b=6,\ n=3v)$. It is easy to see that each treatment occurs in $r=3$ blocks, which are the same size $k=2w$. The information matrix of design d^x has the following form:

$$\mathbf{C}_{\mathrm{d}^x} = 3\mathbf{I}_v - \frac{1}{2w}\mathbf{N}\mathbf{N}' =$$
$$= \begin{pmatrix} 3\mathbf{I}_w - \frac{3}{2w}\mathbf{1}_w\mathbf{1}_w' & \mathbf{0}_{ww} & -\frac{1}{w}\mathbf{1}_w\mathbf{1}_w' & -\frac{1}{2w}\mathbf{1}_w\mathbf{1}_w' \\ \mathbf{0}_{ww} & 3\mathbf{I}_w - \frac{3}{2w}\mathbf{1}_w\mathbf{1}_w' & -\frac{1}{2w}\mathbf{1}_w\mathbf{1}_w' & -\frac{1}{w}\mathbf{1}_w\mathbf{1}_w' \\ -\frac{1}{w}\mathbf{1}_w\mathbf{1}_w' & -\frac{1}{2w}\mathbf{1}_w\mathbf{1}_w' & 3\mathbf{I}_w - \frac{3}{2w}\mathbf{1}_w\mathbf{1}_w' & \mathbf{0}_{ww} \\ -\frac{1}{2w}\mathbf{1}_w\mathbf{1}_w' & -\frac{1}{w}\mathbf{1}_w\mathbf{1}_w' & \mathbf{0}_{ww} & 3\mathbf{I}_w - \frac{3}{2w}\mathbf{1}_w\mathbf{1}_w' \end{pmatrix}. \tag{2.2}$$

The eigenvalues of the information matrix $\mathbf{C}_{\mathrm{d}^x}$ are $\gamma_0=0$, $\gamma_{1^x}=1$, $\gamma_{2^x}=2$ and $\gamma_{3^x}=3$ for any v and $w>1$, with multiplicities 1, 1, 1 and $(v-3)$, respectively. So the $v-1$ dimensional vector of non–zero eigenvalues of matrix (2.2) is denoted $\boldsymbol{\gamma}_{\mathrm{d}^x} = [\gamma_{1^x}, \gamma_{2^x}, \gamma_{3^x}, ..., \gamma_{3^x}]$, and the smallest non–zero eigenvalue for d^x, denoted as γ_{1^x}, is equal to 1.

Let us also consider a binary block design in class $D(v=4w,\ b=6,\ n=3v)$ such that $v_1=w$ treatments have the maximum replication of 6. Let $\mathbf{r}=\mathbf{N1}$ be the treatment replication vector, $\mathbf{k}=\mathbf{N'1}$ be the block size vector, and $n=\mathbf{r'1}=\mathbf{k'1}=3v$ be the number of experimental units, and let $v=4w$ treatments be arranged on $n=3v$ experimental units in the following way:

B_1:	1	2	…	w	…	$2w$	…	$3w$	…	$4w$
B_2:	1	2	…	w	…	$2w$	…	$3w$	…	$4w$
B_3:	1	2	…	w						
B_4:	1	2	…	w						
B_5:	1	2	…	w						
B_6:	1	2	…	w.						

Usually, to study the properties of this block design the following information matrix is used:

$$\mathbf{C}_{d^0} = \mathbf{R} - \mathbf{N}\mathbf{K}^{-1}\mathbf{N}' =$$

$$= \begin{pmatrix} 6\mathbf{I}_w - \frac{9}{2w}\mathbf{1}_w\mathbf{1}_w' & -\frac{1}{2w}\mathbf{1}_w\mathbf{1}_w' & -\frac{1}{2w}\mathbf{1}_w\mathbf{1}_w' & -\frac{1}{2w}\mathbf{1}_w\mathbf{1}_w' \\ -\frac{1}{2w}\mathbf{1}_w\mathbf{1}_w' & 2\mathbf{I}_w - \frac{1}{2w}\mathbf{1}_w\mathbf{1}_w' & -\frac{1}{2w}\mathbf{1}_w\mathbf{1}_w' & -\frac{1}{2w}\mathbf{1}_w\mathbf{1}_w' \\ -\frac{1}{2w}\mathbf{1}_w\mathbf{1}_w' & -\frac{1}{2w}\mathbf{1}_w\mathbf{1}_w' & 2\mathbf{I}_w - \frac{1}{2w}\mathbf{1}_w\mathbf{1}_w' & -\frac{1}{2w}\mathbf{1}_w\mathbf{1}_w' \\ -\frac{1}{2w}\mathbf{1}_w\mathbf{1}_w' & -\frac{1}{2w}\mathbf{1}_w\mathbf{1}_w' & -\frac{1}{2w}\mathbf{1}_w\mathbf{1}_w' & 2\mathbf{I}_w - \frac{1}{2w}\mathbf{1}_w\mathbf{1}_w' \end{pmatrix}, \quad (2.3)$$

where **R** and **K** denote diagonal matrices having successive components of the vectors **r** and **k** respectively as their diagonal elements, and $\mathbf{K}^{-1}$ denotes the inverse of the matrix **K**. Its eigenvalues are $\gamma_0=0$, $\gamma_{1^0} = 2$, and $\gamma_{2^0} = 6$ for any v and $w>1$, with multiplicities 1, $v–w$ and $w–1$, respectively. Hence the $v–1$ dimensional vector of non–zero eigenvalues of matrix (2.3) is denoted $\boldsymbol{\gamma}_{d^0} = [\gamma_{1^0},...,\gamma_{1^0},\gamma_{2^0},...,\gamma_{2^0}]$, and the smallest non–zero eigenvalue, denoted as γ_{1^0}, is equal to 2.

3. Optimality of block designs

The theory of optimal experimental designs is concerned with the problem of selecting a design which minimizes some functional of the matrix $\mathbf{C}_d$ over all possible designs in class D. This functional is called an optimality criterion. The commonly used criteria are called A, D and E criteria. These criteria are defined by functionals of the form

$$\sum_{i=1}^{v-1}\gamma_i^{-1},\ \prod_{i=1}^{v-1}\gamma_i^{-1} \text{ and } \{\min(\gamma_1,\gamma_2,...,\gamma_{v-1})\}^{-1}$$

respectively, where γ_i are the non–zero eigenvalues of matrix $\mathbf{C}_d$, i=1,2,...,v–1. Optimality criteria are defined for connected designs, and a broad discussion of the optimality criteria and their history can be found in Shah and Sinha (1989, chapter 3). Some information about optimality of block designs can be found in Kozłowska (1990, 1996, 1999).

Bagchi and Bagchi (2001) introduced the definition that a design $d^{\#}$ is said to be better than another design $d\in D$ in the sense of majorization (M–better) if the vector $\boldsymbol{\gamma}_{d\#}$ is weakly majorized by the vector $\boldsymbol{\gamma}_d$, and a design $d^{\#}\in D$ is said to be optimal in the sense of majorization (M–optimal) if it is M–better than any other design in *D*.

Hence for any block design we can consider the optimality of the design, particularly when only the bottom stratum is used. We recall some definitions (Marshall and Olkin, 1979).

For any row vector $\mathbf{x} = [x_1, x_2, \ldots, x_v] \in \mathrm{R}^v$ for which x_1 indicates the smallest element in **x**, x_2 indicates the second–smallest element, and so on, we have :

Definition 1. The vector **x** is said to be majorized by the vector **y**, denoted as $\mathbf{x} \prec \mathbf{y}$, if for k=1, 2, …,$v$–1

$$\sum_{i=1}^{\mathrm{k}} x_i \geq \sum_{i=1}^{\mathrm{k}} y_i \quad \text{and} \quad \sum_{i=1}^{v} x_i = \sum_{i=1}^{v} y_i .$$

Majorization is a partial ordering among vectors, which applies to vectors having the same sum. The above conditions may be formulated as follows:

$$\sum_{i=1}^{\mathrm{k}} x_{v+1-i} \leq \sum_{i=1}^{\mathrm{k}} y_{v+1-i} \text{, for k=1, 2, …,} v \text{ and } \sum_{i=1}^{v} x_i = \sum_{i=1}^{v} y_i .$$

Definition 2. The vector **x** is said to be weakly majorized by the vector **y**, denoted as $\mathbf{x} \prec^w \mathbf{y}$, if for k=1, 2, …,$v$–1

$$\sum_{i=1}^{\mathrm{k}} x_i \geq \sum_{i=1}^{\mathrm{k}} y_i . \tag{3.1}$$

Motivated by the above definition we recall (see Bagchi and Bagchi, 2001) the following.

Definition 3. A design $\mathrm{d}^\#$ is said to be better than another design $\mathrm{d} \in D$ in the sense of majorization (M–better) if $\boldsymbol{\gamma}_{\mathrm{d}^\#} \prec^w \boldsymbol{\gamma}_{\mathrm{d}}$.

4. M–better block designs

It can be shown that if $d^{\#}$ is M–better than d then $d^{\#}$ is better than d with respect to all criteria of Type I (see Bagchi and Bagchi, 2001). In particular, it can be shown that it implies E, A and D–optimality.

Theorem. A block design d^{*} is M–better than d^{x} and d^{0} in the class $D(v=4w, b=6, n=3v)$.

Proof. The block designs d^{*}, d^{x} and d^{0} described in section 2 are connected. The property for d^{*} is implied for example by the fact that the number of blocks in which any two treatments being first or second associates occur together is not equal to zero; we have $\lambda_1=3$ and $\lambda_2=1$.

Let us consider whether there is a balanced incomplete block design. For parameters $v=4w$, $b=6$ and $n=3v$ we know that there exists a balanced incomplete block design d^{o} if and only if for some positive scalar s the matrix $\frac{1}{k^{o}}\mathbf{N}^{o}\mathbf{N}^{o'} - s\mathbf{1}_v\mathbf{1}_v'$ is a diagonal matrix, where $k^{o}=v/2=2w$ (Baksalary et al. 1980). Because $n=3v$, hence $r^{o}=3$. It is easy to see that for $w>1$ and any v, the value of $3(v–2)/2(v–1)$ is not a integer. Hence in the binary class $D(v=4w, b=6, n=3v)$ a balanced incomplete block design does not exist.

Consider now the structure of designs d^{*}, d^{x} and d^{0}. For the information matrices $\mathbf{C}_{d^*}$, $\mathbf{C}_{d^x}$ and $\mathbf{C}_{d^0}$ we have $v–1$ dimensional vectors of non–zero eigenvalues of the forms, $\boldsymbol{\gamma}_{d^*} = [2,2,2,3,\ldots,3]$, $\boldsymbol{\gamma}_{d^x} = [1,2,3,\ldots,3]$, and $\boldsymbol{\gamma}_{d^0} = [2,\ldots,2,6,\ldots,6]$, and $\mathrm{tr}(\mathbf{C}_{d^*})=\mathrm{tr}(\mathbf{C}_{d^x})=\mathrm{tr}(\mathbf{C}_{d^0})=6(2w–1)$. It is easy to see that $\boldsymbol{\gamma}_{d^*} \prec^{w} \boldsymbol{\gamma}_{d^x}$, which means that the design d^{*} is M–better than the design d^{x}. Let us consider the relation between vectors $\boldsymbol{\gamma}_{d^*}$ and $\boldsymbol{\gamma}_{d^0}$. Because $w>1$, for k=1, 2, 3 inequality (3.1) holds. Since $\boldsymbol{\gamma}_{d^*} \prec^{w} \boldsymbol{\gamma}_{d^0}$, then the design d^{*} is M–better than the design d^{0}. The theorem holds.

5. Example

Let $w=2$ and d_2^{*}, d_2^{x}, $d_2^{0} \in D(v=8, b=6, n=24)$ be a binary block design with an incidence matrix in the following form:

$$\mathbf{N}_{d_2^*}=\begin{pmatrix}1&1&1&0&0&0\\1&1&1&0&0&0\\1&0&0&1&1&0\\1&0&0&1&1&0\\0&1&0&1&0&1\\0&1&0&1&0&1\\0&0&1&0&1&1\\0&0&1&0&1&1\end{pmatrix},\ \mathbf{N}_{d_2^x}=\begin{pmatrix}1&1&1&0&0&0\\1&1&1&0&0&0\\0&0&0&1&1&1\\0&0&0&1&1&1\\1&1&0&1&0&0\\1&1&0&1&0&0\\0&0&1&0&1&1\\0&0&1&0&1&1\end{pmatrix},$$

$$\mathbf{N}_{d_2^0}=\begin{pmatrix}1&1&1&1&1&1\\1&1&1&1&1&1\\1&1&0&0&0&0\\1&1&0&0&0&0\\1&1&0&0&0&0\\1&1&0&0&0&0\\1&1&0&0&0&0\\1&1&0&0&0&0\end{pmatrix}.$$

The corresponding information matrices are positive semi–definite with rank $v-1=7$. The information matrices have the forms

$$\mathbf{C}_{d_2^*}=\frac{1}{4}\begin{pmatrix}9&-3&-1&-1&-1&-1&-1&-1\\-3&9&-1&-1&-1&-1&-1&-1\\-1&-1&9&-3&-1&-1&-1&-1\\-1&-1&-3&9&-1&-1&-1&-1\\-1&-1&-1&-1&9&-3&-1&-1\\-1&-1&-1&-1&-3&9&-1&-1\\-1&-1&-1&-1&-1&-1&9&-3\\-1&-1&-1&-1&-1&-1&-3&9\end{pmatrix},$$

$$\mathbf{C}_{d_2^x}=\frac{1}{4}\begin{pmatrix}9&-3&0&0&-2&-2&-1&-1\\-3&9&0&0&-2&-2&-1&-1\\0&0&9&-3&-1&-1&-2&-2\\0&0&-3&9&-3&-1&-2&-2\\-2&-2&-1&-3&9&-3&0&0\\-2&-2&-1&-1&-3&9&0&0\\-1&-1&-2&-2&0&0&9&-3\\-1&-1&-2&-2&0&0&-3&9\end{pmatrix},\ \mathbf{C}_{d_2^0}=\frac{1}{4}\begin{pmatrix}15&-9&-1&-1&-1&-1&-1&-1\\-9&15&-1&-1&-1&-1&-1&-1\\-1&-1&7&-1&-1&-1&-1&-1\\-1&-1&-1&7&-1&-1&-1&-1\\-1&-1&-1&-1&7&-1&-1&-1\\-1&-1&-1&-1&-1&7&-1&-1\\-1&-1&-1&-1&-1&-1&7&-1\\-1&-1&-1&-1&-1&-1&-1&7\end{pmatrix}.$$

and have vectors of eigenvalues $\boldsymbol{\gamma}_{d^*}=[2,2,2,3,3,3,3]$, $\boldsymbol{\gamma}_{d^x}=[1,2,3,3,3,3,3]$, $\boldsymbol{\gamma}_{d^0}=[2,2,2,2,2,2,6]$, respectively. It is easy to see that $\boldsymbol{\gamma}_{d^*}\prec^{w}\boldsymbol{\gamma}_{d^x}$ and

$\gamma_{d^*} \prec^{w} \gamma_{d^0}$. Hence design d^* is M–better than design d^x and is M–better than design d^0.

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