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ON THE LÖWNER ORDERING OF C – MATRICES OF BTIB DESIGNS

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Summary

The aim of the present paper is to establish a counterpart of Theorem 4.1 in Kozłowska et al. (2013) for balanced treatment incomplete block designs. Compared with the aforementioned result, an improvement is achieved by giving conditions that are both necessary and sufficient for the corresponding analogue of Theorem 4.1 to hold.

Keywords and phrases: balanced treatment incomplete block design, Löwner ordering, M–optimality criterion

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1. Preliminaries

Our aim here is to announce an analogue of a recent result due to Kozłowska et al. (2013, Theorem 4.1) for balanced treatment incomplete block designs. To this end, we consider proper block designs for comparing a set of test treatments with a control treatment. We assume that v test treatments labelled $1,\ldots,v$ have to be compared with the control labelled 0 in b blocks each of size k, where $2 \le k \le v$.

The following class of treatment-control designs was originally considered by Bechhofer and Tamhane (1981), see also Hinkelmann and Kempthorn (2005, 6.5.3).

Definition. A block design with v test treatments and one control treatment in b blocks each of size $1 < k < v+1$ is called a *balanced treatment incomplete block design*, denoted by $\mathrm{BTIBD}(v, b, k, \lambda_0, \lambda_1)$, if

1. each test treatment occurs together with the control λ_0 times in a block,
2. any two test treatments occur together λ_1 times in a block.

Recall that the combinatorial structure of a BTIBD is given by its $(v+1)\times b$ incidence matrix $\boldsymbol{N}$ whose entries n_{ij} give the number of times treatment i ($i = 0,1,\ldots,v$) occurs in block j ($j = 1,\ldots,b$).

We are interested in pointing out the following properties coming from the definition of a BTIBD. The concurrence matrix of a BTIBD is of the form

$$\boldsymbol{NN}^T = \begin{pmatrix} r_0 & \lambda_0 & \cdots & \cdots & \lambda_0 \\ \lambda_0 & r_1 & \lambda_1 & \cdots & \lambda_1 \\ \vdots & \lambda_1 & r_2 & \cdots & \vdots \\ \vdots & \cdots & \cdots & \ddots & \lambda_1 \\ \lambda_0 & \lambda_1 & \cdots & \lambda_1 & r_v \end{pmatrix},$$

where $\boldsymbol{N}^T$ denotes the transpose of $\boldsymbol{N}$.

Further, it follows that the self-concurrences r_0 and r_i ($i = 1,\ldots,v$) satisfy the following conditions

$$r_0 k = r_0 + \lambda_0 v, \quad r_i k = r_i + \lambda_0 + \lambda_1(v-1).$$

Hence

$$r_0 - k^{-1} r_0 = k^{-1}\lambda_0 v, \quad r_i - k^{-1} r_i = k^{-1}(\lambda_0 + \lambda_1(v-1)).$$

If we write $a = k^{-1}\lambda_0$, $b = k^{-1}\lambda_1$, then the usual information matrix for the treatment effects of a BTIBD is given by

$$\mathbf{C} = \begin{pmatrix} va & -a\mathbf{1}^T \\ -a\mathbf{1} & (a+bv)\mathbf{I} - b\mathbf{J} \end{pmatrix} = \begin{pmatrix} va & -a\mathbf{1}^T \\ -a\mathbf{1} & a\mathbf{I} + bv(\mathbf{I} - v^{-1}\mathbf{J}) \end{pmatrix}, \tag{1.1}$$

where $\mathbf{1}$ is the column-vector of v ones and $\mathbf{J} = \mathbf{1}\mathbf{1}^T$. It is a notable feature of a BTIBD that it has a supplemented balance, that is the principal minor of $\mathbf{C}$ formed by deleting the row and the column corresponding to the control is completely symmetric.

2. M – criterion under the Löwner ordering – a characterization of BTIBDs

Following Kozłowska et al. (2013), M – optimality criterion, introduced by Bagchi and Bagchi (2001), will be considered here for assessing BTIBDs.

Given a symmetric matrix $\mathbf{C}$ of order n, let $\mu(\mathbf{C}) = (\mu_1(\mathbf{C}), \ldots, \mu_n(\mathbf{C}))$ be the vector of eigenvalues of $\mathbf{C}$ with $\mu_1(\mathbf{C}) \leq \ldots \leq \mu_n(\mathbf{C})$. Let $\mathbf{C}_1$ and $\mathbf{C}_2$ be the information matrices of block designs d_1 and d_2, respectively.

A design d_1 is said to be M – *better t*han a design d_2 if $\mu(\mathbf{C}_1)$ is weakly majorized from above *b*y $\mu(\mathbf{C}_2)$ in the following sense

$$\mu(\mathbf{C}_1) \prec^w \mu(\mathbf{C}_2) \Leftrightarrow \sum_{i=1}^{\ell} \mu_i(\mathbf{C}_1) \geq \sum_{i=1}^{\ell} \mu_i(\mathbf{C}_2), \quad \ell = 1, \ldots, n.$$

Furthermore, we write $\mathbf{C}_1 \prec^L \mathbf{C}_2$ if $\mathbf{C}_2 - \mathbf{C}_1$ is positive semidefinite. The relation $\prec^L$ is known as the Löwner ordering of symmetric matrices.

The following implication is well known in the literature, see e.g. Marshall et al. (2011, page 360).

$$\mathbf{C}_1 \prec^L \mathbf{C}_2 \Rightarrow \mu(\mathbf{C}_2) \prec^w \mu(\mathbf{C}_1).$$

This fact makes it interesting to give convenient criteria for the relation $\mathbf{C}_1 \prec^L \mathbf{C}_2$ to hold. Theorem 4.1 in Kozłowska et al. (2013) gives a sufficient condition for $\mathbf{C}$ – matrices of S type block designs to be in the Löwner partial ordering. Our goal is to come up with the necessary and sufficient conditions for ensuring $\mathbf{C}_1 \prec^L \mathbf{C}_2$ for BTIBDs.

To this end, let $\mathbf{C}_1$ and $\mathbf{C}_2$ be two information matrices of type (1.1) with parameters a_1, b_1 and a_2, b_2, respectively, and let $\alpha = a_2 - a_1$, $\beta = b_2 - b_1$. Then

$$\mathbf{C}_2 - \mathbf{C}_1 = \begin{pmatrix} v\alpha & -\alpha\mathbf{1}^T \\ -\alpha\mathbf{1} & (\alpha + v\beta)\mathbf{I} - \beta\mathbf{J} \end{pmatrix} = \begin{pmatrix} v\alpha & -\alpha\mathbf{1}^T \\ -\alpha\mathbf{1} & \alpha\mathbf{I} + v\beta\left(\mathbf{I} - v^{-1}\mathbf{J}\right) \end{pmatrix}. \quad (2.1)$$

We will make use of the following fact (Pukelsheim, 1993, 3.12).

Lemma. A symmetric block matrix

$$\begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix}$$

is positive semidefinite if and only if $\mathbf{A}_{11}$ is positive semidefinite, the range of $\mathbf{A}_{11}$ includes the range of $\mathbf{A}_{12}$, and the Schur complement

$$\mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-}\mathbf{A}_{12}$$

is positive semidefinite.

The main result of this paper is given below.

Theorem. Under the assumptions above, $\mathbf{C}_2 - \mathbf{C}_1$ is positive semidefinite if and only if

$$\alpha \geq 0 \quad \textit{and} \quad \alpha + v\beta \geq 0.$$

Proof. By referring to (2.1) and the lemma above, we see that $\mathbf{C}_2 - \mathbf{C}_1$ is positive semidefinite if and only if $\alpha \geq 0$ and the Schur complement

$$\mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-}\mathbf{A}_{12} = (\alpha + v\beta)\mathbf{I} - \beta\mathbf{J} - \alpha^2\mathbf{J}(v\alpha)^{-1} = (\alpha + v\beta)\left(\mathbf{I} - v^{-1}\mathbf{J}\right)$$

is positive semidefinite. Since $\mathbf{Q} = \mathbf{I} - v^{-1}\mathbf{J}$ is an oblique projector, $\alpha + v\beta \geq 0$ is necessary and sufficient for $(\alpha + v\beta)\mathbf{Q}$ to be positive semidefinite. Hence the result holds.

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