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STRATUM ANALYSES FOR SPLIT-SPLIT-PLOT DESIGNS GENERATED BY GROUP DIVISIBLE DESIGNS

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Summary

We present general and particular analyses under a mixed linear model resulting from the construction of a non-orthogonal split-split-plot (SSP) design and proper randomization performed (see, Ambroży and Mejza, 2013). The design was generated by some group divisible block designs. Attention is paid to optimal statistical properties with respect to the efficiency of estimation of some groups of basic and any contrasts in both the generating and final designs. The considerations are illustrated with simulated data from an experiment with grain yields of winter wheat.

Key words and phrases: any treatment contrast, basic treatment contrast, general balance, partial efficiency, split-split-plot design, stratum analyses

Classification AMS 2010: 62K10, 62K15

1. Introduction

The aim of this paper is to present general and particular analyses under a mixed linear model resulting from a construction of non-orthogonal split-split-plot (SSP) design and proper randomization performed as it is given by Ambroży and Mejza (2013). This work is a continuation of the mentioned above paper. Given there method of the construction is based on Kronecker's product of three matrices, $\mathbf{N}_A, \mathbf{N}_B$ and $\mathbf{N}_C$. Two first of them are incidence matrices of generating designs for factors *A* and *B* while the third one is an incidence matrix of a randomized complete block (RBD) design (for factor *C*). To reduce a number of blocks in the final design it can be assumed that $\mathbf{N}_C$ is reduced to one block only ($\mathbf{N}_C = \mathbf{1}$).

To illustrate statistical analyses proposed in the present paper some simulated data from an original experiment will be used. It means that conclusions reached here should be treated as a category of a methodology for planning and analysis such experiments. In these analyses, both basic (Pearce et al., 1974) and any treatment contrasts (e.g. Ceranka and Mejza, 1979) will play an essential role. R and *STATISTICA* packages were used for all calculations.

2. Stratum analyses

2.1. Multistratum model

In the multistratum SSP type experiment the stratum analyses are based on submodels $\mathbf{y}_f = \mathbf{P}_f \mathbf{y}$, f = 0, 1, 2, 3, 4, where y is an $n \times 1$ vector of the lexicographically ordered observations with $\mathbf{P}_f$ being orthogonal projection on the *f*-th stratum and *f* denotes the number of the stratum i.e. the total-area stratum ("zero" stratum), the inter-block stratum (1), the inter-whole plot stratum (2), the inter-subplot stratum (3) and the inter-sub-subplot stratum (4). One can find forms of the matrices $\mathbf{P}_f$, *f* = 0, 1, 2, 3, 4, in Mejza (1997).

The assumed orthogonal block structure of the considered SSP design allows one to apply Nelder's (1965a, 1965b) approach to the analysis of variance, where stratum information matrices for the treatment combinations, $\mathbf{A}_f = \mathbf{\Delta} \mathbf{P}_f \mathbf{\Delta}'$ play important role wherein $\mathbf{\Delta}'$ is an $n \times v$ design matrix for *v* treatment combinations. One can find the forms of the $\mathbf{A}_f$, *f* = 0, 1, 2, 3, 4, in Ambroży and Mejza (2013).

The matrix $\mathbf{A}_0$ in "zero" stratum is suitable mainly for the estimating the general mean of the experiment. The algebraic properties of the other $\mathbf{A}_f$, *f* = 1, 2, 3, 4, are strictly connected with statistical properties of the SSP design, in particular with estimability of linear functions of treatment parameters. From the fact (cf. Ambrozy and Mejza, 2013) that $\mathbf{A}_f \mathbf{1} = \mathbf{0}$, f = 1, 2, 3, 4, it follows that if a linear function $\mathbf{c}'\boldsymbol{\tau}$ is estimable in the f-th stratum then it must be a contrast, i.e. $\mathbf{c}'\mathbf{1} = 0$, wherein $\boldsymbol{\tau}$ ($v \times 1$) is the vector of fixed treatment combination effects.

Describing statistical properties of the SSP design and performing the stratum analyses will be highly simplified if we will use basic treatment combination contrasts under assumption that the design is generally balanced.

2.2. Basic contrasts with reference to property of general balance

Let ε_{fh} $(0 \le \varepsilon_{fh} \le 1)$ be an eigenvalue of the matrix $\mathbf{A}_f$ with respect to $\mathbf{r}^\delta = \mathrm{diag}(r_1, r_2, ..., r_v)$ and $\mathbf{p}_h$ be the corresponding eigenvector, where f = 0, 1, 2, 3, 4, r_h is a replicate of the h-th treatment combination with h = 1, 2,..., v The eigenvectors can be chosen to be mutually $\mathbf{r}^\delta$-orthonormal, i.e. $\mathbf{p}'_h \mathbf{r}^\delta \mathbf{p}_h = 1$ and $\mathbf{p}'_h \mathbf{r}^\delta \mathbf{p}_{h'} = 0$, for $h \ne h'$, where h, h' = 1, 2,.., v. Since $\mathbf{A}_f \mathbf{1}_v = \mathbf{0}$, $f > 0$, the last eigenvector $\mathbf{p}_v$ may be chosen as $n^{-1/2}\mathbf{1}_v$.

Let us note that $\mathbf{c}_h = \mathbf{r}^\delta \mathbf{p}_h$ define (basic) treatment combination contrasts of the form $\mathbf{c}'_h \tau$, h = 1, 2,..., v-1 (cf. Pearce et al., 1974). The contrast $\mathbf{c}'_h \tau$ is estimable in the f-th stratum, f = 0, 1, 2, 3, 4, when the following relation holds

$$\mathbf{A}_f \mathbf{p}_{fh} = \varepsilon_{fh} \mathbf{r}^\delta \mathbf{p}_{fh}, \tag{2.1}$$

$$\varepsilon_{fh} \ne 0, \quad f = 0, 1, 2, 3, 4; \quad h = 1, 2, ..., v\text{-}1.$$

Then the eigenvalue ε_{fh} can be identified as a stratum efficiency factor of the SSP design with respect to the estimation of the basic contrast $\mathbf{c}'_h \tau$ in the f-th stratum. All considerations are simplified if the information matrices $\mathbf{A}_f$ have a common set of eigenvectors with respect to $\mathbf{r}^\delta$, i.e. the matrices $\mathbf{A}_f$ mutually commute with respect to $\mathbf{r}^{-\delta}$: $\mathbf{A}_f \mathbf{r}^{-\delta} \mathbf{A}_{f'} = \mathbf{A}_{f'} \mathbf{r}^{-\delta} \mathbf{A}_f$, for f, f' = 0, 1, 2, 3, 4; $f \ne f'$ and $\mathbf{r}^{-\delta} = \mathrm{diag}(1/r_1, 1/r_2, \ldots, 1/r_v)$.

When those conditions hold, the SSP design considered is generally balanced designs (Mejza, 1992) and then, the eigenvalues ε_{fh} satisfy relation

$\varepsilon_{1h} + \varepsilon_{2h} + \varepsilon_{3h} + \varepsilon_{4h} = 1$, for $h < v$. The common eigenvectors of $\mathbf{A}_f$ (f = 0, 1, 2, 3, 4), with respect to $\mathbf{r}^{\delta}$ can be generated by different ways. One of the methods will be illustrated in the Section 3 (Results). The stratum BLUE of $\mathbf{c}'_h\tau$ can be obtained as

$$(\widehat{\mathbf{c}'_h\boldsymbol{\tau}})_f = \varepsilon_{fh}^{-1}\mathbf{p}'_h\mathbf{Q}_f, \quad \text{where } \mathbf{Q}_f = \Delta\mathbf{P}_f\mathbf{y}$$

with

$$\mathrm{Var}\{(\widehat{\mathbf{c}'_h\boldsymbol{\tau}})_f\} = \varepsilon_{fh}^{-1}\gamma_f, \quad f = 0, 1, 2, 3, 4; \; h = 1, 2, ..., v\text{-}1,$$

where $\gamma_f \geq 0$ are stratum variances.

2.3. Any treatment combination contrasts

In the experiments we are often interested in an estimation and testing hypothesis concerning any treatment contrasts different than basic contrasts. Let $\mathbf{s}'\boldsymbol{\tau}$ be a contrast. It is easily seen that $\mathbf{s}'\boldsymbol{\tau}$ is estimable in the f-th stratum iff it can be written as

$$\mathbf{s} = \lambda_{f1}\mathbf{c}_1 + \lambda_{f2}\mathbf{c}_2 + ... + \lambda_{fh}\mathbf{c}_h,$$

where λ_{fj}, f = 1, 2, 3, 4; j = 1, 2,..., h, are scalars such that $\lambda_{f1}^2 + \lambda_{f2}^2 + ... + \lambda_{fh}^2 > 0$ and h denotes the number of the basic contrasts estimable in the f-th stratum. Since vectors $\mathbf{c}_j$, j = 1, 2,..., h, are mutually $\mathbf{r}^{-\delta}$-orthonormal, then (cf. Ceranka and Mejza, 1979)

$$\lambda_{fj} = \mathbf{c}'_j\mathbf{r}^{-\delta}\mathbf{s} = \mathbf{p}'_j\mathbf{s}, \qquad f = 1, 2, 3, 4; \; j = 1, 2, ..., h.$$

Hence a stratum estimator of the contrast $\mathbf{s}'\boldsymbol{\tau}$ has the form

$$(\widehat{\mathbf{s}'\boldsymbol{\tau}})_f = \sum_{j=1}^{h}\lambda_{fj}(\widehat{\mathbf{c}'_j\boldsymbol{\tau}})_f$$

and its variance is the following:

$$\mathrm{Var}[(\hat{\mathbf{s}'\boldsymbol{\tau}})_f] = \gamma_f \sum_{j=1}^{h} \frac{\lambda_{fj}^2}{\varepsilon_{fj}}.$$

The efficiency factor of the design for any treatment contrast estimable in the f-th stratum is given by

$$\mathrm{E}_f(\hat{\mathbf{s}'\boldsymbol{\tau}}) = \left(\sum_{j=1}^{h} \lambda_{fj}^2\right) / \sum_{j=1}^{h} (\lambda_{fj}^2 / \varepsilon_{fj}).$$

2.4. Testing hypotheses

If normality of random variables of the model is assumed, it is easy to construct an exact test of the hypothesis $\mathrm{H}_{0f} : \boldsymbol{\tau}'\mathbf{A}_f\boldsymbol{\tau} = 0$, f = 1, 2, 3, 4, relating to all the treatment (combination) contrasts estimable in the f-th stratum (ANOVA). In particular we are interested in testing hypothesis for any treatment contrasts, $\mathrm{H}_{0f}^* : \mathbf{s}'\boldsymbol{\tau} = 0$, estimable in the f-th stratum. It is easy to express (cf. Graybill, 1961) also general hypothesis, $\mathrm{H}_0 : \mathbf{W}'\boldsymbol{\tau} = \mathbf{0}$ for all basic contrasts estimable in the f-th stratum connected with the main or interaction effects of the factors. The number of these treatment combination contrasts (called shortly "Treatments") is equal to rank of the matrix **W**, i.e. r(**W**). The necessary sum of squares for „Treatments" in ANOVA can be obtained by the formula

$$\mathrm{SST}_f = \sum_h \varepsilon_{fh} [(\hat{\mathbf{c}_h'\boldsymbol{\tau}})_f]^2, \qquad f = 1, 2, 3, 4,$$

while the sum of squares for errors are as follows

$$\mathrm{SSE}_f = \mathrm{SSY}_f - \mathrm{SST}_f, \text{ where } \mathrm{SSY}_f = \mathbf{y}'\mathbf{P}_f\mathbf{y}.$$

These sums are sufficient to build the appropriate F-tests.

Table 1. ANOVA in the f-th stratum, f = 1, 2, 3, 4.

Source of variation	DF	SS	E(MS)
„Treatments" (f)	$\nu_{Tf} = \mathrm{r}(\mathbf{A}_f)$	SST_f	$\gamma_f + (\nu_{Tf})^{-1}\tau'\mathbf{A}_f\tau$
Error (f)	$\nu_{Ef} = \nu_f - \nu_{Tf}$	SSE_f	γ_f
Total (f)	$\nu_f = \mathrm{r}(\mathbf{P}_f)$	SSY_f	$\gamma_f + (\nu_f)^{-1}\tau'\mathbf{A}_f\tau$

3. Results

Let us consider an $s \times t \times w$ experiment designed to investigate a response of yield in $s = 7$ varieties (genotypes) of winter wheat for $t = 5$ different doses of nitrogen fertilization and a chemical preparation – growth regulator ($w = 2$). The experiment was carried out in a split-split-plot (SSP) design incomplete with respect to whole plot treatments (the varieties) and also subplot treatments (nitrogen fertilizations) while complete with respect to sub-subplot treatments (the growth regulator). The method of construction of the SSP design was presented by Ambroży and Mejza (2013).

As was mentioned in that paper the genotypes comprised six new varieties ($s_1 = 6$; $A_1, ..., A_6$) called the test whole plot (A) treatments and one standard variety (A_7) called the standard whole plot (A) treatment. The test subplot (B) treatments were defined by increasing fertilization doses: B_1, B_2, B_3, B_4 ($t_1 = 4$), and B_5 (no fertilization) signified the standard (control) subplot treatment. The sub-subplot treatments corresponded to the application (or no application) of the chemical preparation: C_1, C_2 ($w = 2$).

Because of the fact that an experimental material connected with new varieties was limited, this experiment was conducted in an incomplete SSP design with an incidence matrix $\mathbf{N}_1 = \mathbf{N}_A \otimes \mathbf{N}_B \otimes \mathbf{1}_w$, where $\mathbf{N}_A$ and $\mathbf{N}_B$ are as follows:

$$\mathbf{N}_A = \begin{bmatrix} \tilde{\mathbf{N}}_A \\ \mathbf{1}'_3 \end{bmatrix}, \qquad \mathbf{N}_B = \begin{bmatrix} \tilde{\mathbf{N}}_B \\ \mathbf{1}'_4 \end{bmatrix}$$

wherein $\tilde{\mathbf{N}}_A$ and $\tilde{\mathbf{N}}_B$ are incidence matrices of group divisible block designs of types S1 and SR1, respectively (Clatworthy, 1973). According to this fact let us assume the test A treatments and the test B treatments can be grouped with regard to the following S1 and SR1 associate schemes, respectively:

Groups	G_{21}	G_{22}
G_{11}	A_1	A_4
G_{12}	A_2	A_5
G_{13}	A_3	A_6

Groups	G_{21}	G_{22}
G_{11}	B_1	B_3
G_{12}	B_2	B_4

The corresponding incidence matrices $\tilde{\mathbf{N}}_A$ and $\tilde{\mathbf{N}}_B$ are given by Ambroży and Mejza (2013). The generated SSP design with the incidence matrix $\mathbf{N}_1$ has the following parameters:

$$v = 70, \quad b = 12, \quad k = 30, \quad \mathbf{r} = [2, 2, 2, 2, 2, 2, 3]' \otimes [2, 2, 2, 2, 4]' \otimes \mathbf{1}_2,$$

where v, b, k are the number of treatment combinations, the number of blocks, the number of units within the blocks, respectively and $\mathbf{r}$ denotes the vector of replicates of the treatment combinations.

Statistical analysis started with the appointment of the information matrices $\mathbf{A}_f$, f = 1, 2, 3, 4, according to the formula given in Section 2.2. Then we investigated algebraic properties of them. It should be noted that we can obtain eigenvalues ε_{fh} of $\mathbf{A}_f$ (stratum efficiency factors) using the eigenvalues of the generating matrices (Ambroży and Mejza, 2013: Table 1) or by an appropriate computer program (here R package). In both cases we need a set of eigenvectors (mutually commuted with respect to $\mathbf{r}^{-\delta}$) which generate the basic contrasts (see, Section 2.2), where $\mathbf{r}^{-\delta} = \begin{bmatrix} (1/2)\mathbf{I}_6 & \mathbf{0} \\ \mathbf{0} & 1/3 \end{bmatrix} \otimes \begin{bmatrix} (1/2)\mathbf{I}_4 & \mathbf{0} \\ \mathbf{0} & 1/4 \end{bmatrix} \otimes \mathbf{I}_2$.

The computer programs create always a set of eigenvectors, which usually does not have interpretations according to the aim of the experiment.

In the paper we propose a procedure of creating the set of basic contrasts using information relating to the generating S1 and SR1 designs. In this kind of the experiment, we are mainly interested in comparisons among the test A treatment effects and among the test B treatment effects. We suppose that the experimenter is also interested in the comparisons between a set of the test A (or B) treatment effects and the control A (or B) treatment effect, respectively. Taking into account the associate schemes S1 and SR1 provided above, let

$\mathbf{g}_{1A} = [1 \quad -1 \quad 0]'$ or $\mathbf{g}_{1A} = [1 \quad 1 \quad -2]'$ (among rows)
and $\mathbf{g}_{2A} = [1 \quad -1]'$ (between columns); $\mathbf{g}_{1B} = [1 \quad -1]'$ (between rows)
and $\mathbf{g}_{2B} = [1 \quad -1]'$ (between columns). (3.1)

The vectors (3.1) create comparisons among test A (B) treatment effects inside groups and among interaction treatment effects from different groups.

They also relate to comparisons connected with test *A* (*B*) treatment effects between groups only. They all are used to build the orthonormal eigenvectors of the information matrices $\tilde{\mathbf{C}}_A$ (Table 2) and $\tilde{\mathbf{C}}_B$ (Table 3). Their corresponding eigenvalues were presented also by Ambroży and Mejza (2013). In turn Tables 4 and 5 present sets of orthonormal eigenvectors of the information matrices $\mathbf{C}_A$ and $\mathbf{C}_B$, respectively. Assuming also that $\mathbf{p}_{C1} = \frac{1}{\sqrt{2}}[1 \quad -1]'$, $\mathbf{p}_{C2} = \frac{1}{\sqrt{2}}[1 \quad 1]'$ we can construct the common set of $\mathbf{r}^{-\delta}$- orthonormal eigenvectors for the SSP design as $\mathbf{p}_h = \mathbf{p}_{Ai} \otimes \mathbf{p}_{Bj} \otimes \mathbf{p}_{Ck}$,

where *i* = 1, 2.…. 7; *j* = 1, 2..... 5; *k* = 1, 2; $h = tw(i-1) + w(j-1) + k$. We can note (see, Section 2.2) that $\mathbf{c}_h = \mathbf{r}^{\delta}\mathbf{p}_h$ defines a (basic) contrast $\mathbf{c}_h'\boldsymbol{\tau}$, *h* = 1, 2,..., 69. The last vector $\mathbf{p}_{70}$ does not define the contrast. In any case, we should check using formula (2.1) from Section 2.2. in which stratum the contrast $\mathbf{c}_h'\boldsymbol{\tau}$ is estimable.

Table 2. Eigenvalues and eigenvectors for the generating subdesign $\tilde{d}_A$ for the test *A* treatments

Eigenvalues of matrix $\tilde{\mathbf{C}}_A$	Common eigenvectors
$\tilde{\varepsilon}_2^A = 0.75$	$\tilde{\mathbf{p}}_{A3} = \mathbf{1}_2 \otimes \mathbf{g}_{1A} = \frac{1}{2}[1 \quad -1 \quad 0 \quad 1 \quad -1 \quad 0]'$ $\tilde{\mathbf{p}}_{A4} = \mathbf{1}_2 \otimes \mathbf{g}_{1A} = \frac{1}{\sqrt{12}}[1 \quad 1 \quad -2 \quad 1 \quad 1 \quad -2]'$
$\tilde{\varepsilon}_1^A = 1$	$\tilde{\mathbf{p}}_{A1} = \mathbf{g}_{2A} \otimes \mathbf{1}_3 = \frac{1}{\sqrt{6}}[1 \quad 1 \quad 1 \quad -1 \quad -1 \quad -1]'$ $\tilde{\mathbf{p}}_{A2} = \mathbf{g}_{2A} \otimes \mathbf{g}_{1A} = \frac{1}{\sqrt{12}}[1 \quad 1 \quad -2 \quad -1 \quad -1 \quad 2]'$ $\tilde{\mathbf{p}}_{A5} = \mathbf{g}_{2A} \otimes \mathbf{g}_{1A} = \frac{1}{2}[1 \quad -1 \quad 0 \quad -1 \quad 1 \quad 0]'$
0	$\tilde{\mathbf{p}}_{A6} = \frac{1}{\sqrt{6}}\mathbf{1}_6$

Table 3. Eigenvalues and eigenvectors for the generating subdesign $\tilde{d}_B$ for the test B treatments

Eigenvalues of matrix $\tilde{\mathbf{C}}_B$	Common eigenvectors
$\tilde{\varepsilon}_2^B = 0.5$	$\tilde{\mathbf{p}}_{B1} = \mathbf{g}_{1B} \otimes \mathbf{1}_2 = \frac{1}{2}[1 \quad 1 \quad -1 \quad -1]'$ $\tilde{\mathbf{p}}_{B3} = \mathbf{g}_{1B} \otimes \mathbf{g}_{2B} = \frac{1}{2}[1 \quad -1 \quad -1 \quad 1]'$
$\tilde{\varepsilon}_1^B = 1$	$\tilde{\mathbf{p}}_{B2} = \mathbf{1}_2 \otimes \mathbf{g}_{2B} = \frac{1}{2}[1 \quad -1 \quad 1 \quad -1]'$
0	$\tilde{\mathbf{p}}_{B4} = \frac{1}{2}\mathbf{1}_4$

Table 4. Eigenvalues and common eigenvectors for the generating subdesign d_A for the factor A

Eigenvalues of matrix $\mathbf{C}_A$	Common eigenvectors
$\varepsilon_2^A = 0.8$	$\mathbf{p}_{A3} = \frac{1}{\sqrt{2}}[\tilde{\mathbf{p}}_{A3} \quad 0]'$ $\mathbf{p}_{A4} = \frac{1}{\sqrt{2}}[\tilde{\mathbf{p}}_{A4} \quad 0]'$
$\varepsilon_1^A = 1$	$\mathbf{p}_{A1} = \frac{1}{\sqrt{2}}[\tilde{\mathbf{p}}_{A1} \quad 0]'$ $\mathbf{p}_{A2} = \frac{1}{\sqrt{2}}[\tilde{\mathbf{p}}_{A2} \quad 0]'$ $\mathbf{p}_{A5} = \frac{1}{\sqrt{2}}[\tilde{\mathbf{p}}_{A5} \quad 0]'$
$\varepsilon_0^A = 1$	$\mathbf{p}_{A6} = \frac{1}{\sqrt{10}}[\tilde{\mathbf{p}}_{A6} \quad -4]'$
0	$\mathbf{p}_{A7} = \frac{1}{\sqrt{15}}\mathbf{1}_7$

Table 5. Eigenvalues and common eigenvectors for the generating subdesign d_B for the factor *B*

Eigenvalues of matrix $\mathbf{C}_B$	Common eigenvectors
$\varepsilon_2^B = 2/3$	$\mathbf{p}_{B1} = \frac{1}{\sqrt{2}}[\tilde{\mathbf{p}}_{B1} \quad 0]'$ $\mathbf{p}_{B3} = \frac{1}{\sqrt{2}}[\tilde{\mathbf{p}}_{B3} \quad 0]'$
$\varepsilon_1^B = 1$	$\mathbf{p}_{B2} = \frac{1}{\sqrt{2}}[\tilde{\mathbf{p}}_{B2} \quad 0]'$
$\varepsilon_0^B = 1$	$\mathbf{p}_{B4} = \frac{1}{\sqrt{6}}[\tilde{\mathbf{p}}_{B4} \quad -2]'$
0	$\mathbf{p}_{B5} = \frac{1}{\sqrt{12}}\mathbf{1}_5$

The proposed contrasts are strictly connected with the comparisons among the main effects of the considered factors and interaction effects between them. In the present paper, we consider the following types of the basic treatment contrasts (see, Table 6):

- among main effects of the whole plot treatments including: test *A* treatments (A^T) and between the group of test *A* treatments and the whole plot standard (A^T *vs.* A^{SD});
- among main effects of the subplot (*B*) treatments including: test *B* treatments (B^T) and between the group of test *B* treatments and the subplot standard (B^T vs. B^{SD});
- between main effects of the sub-subplot (*C*) treatments;
- other interaction contrasts.

Stratum efficiency factors of the considered SSP design with respect to these contrasts (expressed by the eigenvalues ε_{fh}) are given in parentheses in Table 6 (see also Ambroży and Mejza, 2013: Table 2). Moreover in this table we present estimates of the contrasts with information in which strata they are estimated (see, Section 2.2).

Table 6. Stratum estimates of the basic contrasts (SSP design)

Indexes				Type of contrasts	Strata			
h	j	k	l		(1)	(2)	(3)	(4)
1	1	1	1	$A^T \otimes B^T \otimes C$				-1.7371 (1)
2	1	1	2	$A^T \otimes B^T$		-11.9122 (1/3)	-10.9823 (2/3)	
3	1	2	1	$A^T \otimes B^T \otimes C$				-1.1915 (1)
4	1	2	2	$A^T \otimes B^T$			2.3419 (1)	
5	1	3	1	$A^T \otimes B^T \otimes C$				-1.8786 (1)
6	1	3	2	$A^T \otimes B^T$		-1.9341 (1/3)	-3.2840 (2/3)	
7	1	4	1	$A^T \otimes (B^T vs.B^{SD}) \otimes C$				0.5338 (1)
8	1	4	2	$A^T \otimes (B^T vs.B^{SD})$			-0.4963 (1)	
9	1	5	1	$A^T \otimes C$				3.0140 (1)
10	1	5	2	A^T		8.3774 (1)		
11	2	1	1	$A^T \otimes B^T \otimes C$				0.6899 (1)
12	2	1	2	$A^T \otimes B^T$		12.1923 (1/3	10.1195 (2/3)	
13	2	1	1	$A^T \otimes B^T \otimes C$				0.3562 (1)
14	2	1	2	$A^T \otimes B^T$			6.0758 (1)	
15	2	3	1	$A^T \otimes B^T \otimes C$				-1.9667 (1)
16	2	3	2	$A^T \otimes B^T$		-0.5195 (1/3)	-1.2697 (2/3)	
17	2	4	1	$A^T \otimes (B^T vs.B^{SD}) \otimes C$				-1.4902 (1)
18	2	4	2	$A^T \otimes (B^T vs.B^{SD})$			-6.1253 (1)	
19	2	5	1	$A^T \otimes C$				-0.3262 (1)
20	2	5	2	A^T		-0.4988 (1)		
21	3	1	1	$A^T \otimes B^T \otimes C$				5.3192 (1)
22	3	1	2	$A^T \otimes B^T$	9.0474 (1/15)	9.0165 (4/15)	5.9441 (2/3)	
23	3	2	1	$A^T \otimes B^T \otimes C$				-1.7183 (1)

Indexes				Type of contrasts	Strata			
h	j	k	l		(1)	(2)	(3)	(4)
24	3	2	2	$A^T \otimes B^T$			3.9492 (1)	
25	3	3	1	$A^T \otimes B^T \otimes C$				-1.3046 (1)
26	3	3	2	$A^T \otimes B^T$	-4.0340 (1/15)	-2.8121 (4/15)	-6.3100 (2/3)	
27	3	4	1	$A^T \otimes (B^T vs. B^{SD}) \otimes C$				-1.7820 (1)
28	3	4	2	$A^T \otimes (B^T vs. B^{SD})$			-1.3901 (1)	
29	3	5	1	$A^T \otimes C$				2.2105 (1)
30	3	5	2	A^T	-3.6142 (2/10)	6.7357 (8/10)		
31	4	1	1	$A^T \otimes B^T \otimes C$				0.0143 (1)
32	4	1	2	$A^T \otimes B^T$	4.9623 (1/15)	4.5540 (4/15)	4.3509 (2/3)	
33	4	2	1	$A^T \otimes B^T \otimes C$				-0.0684 (1)
34	4	2	2	$A^T \otimes B^T$			3.1057 (1)	
35	4	3	1	$A^T \otimes B^T \otimes C$				2.5015 (1)
36	4	3	2	$A^T \otimes B^T$	-6.4728 (1/15)	-1.5751 (4/15)	-2.3097 (2/3)	
37	4	4	1	$A^T \otimes (B^T vs. B^{SD}) \otimes C$				-1.6210 (1)
38	4	4	2	$A^T \otimes (B^T vs. B^{SD})$			-11.0350 (1)	
39	4	5	1	$A^T \otimes C$				-0.8287 (1)
40	4	5	2	A^T	-5.6617 (2/10)	-9.2903 (8/10)		
41	5	1	1	$A^T \otimes B^T \otimes C$				1.0801 (1)
42	5	1	2	$A^T \otimes B^T$		1.0536 (1/3)	-1.3780 (2/3)	
43	5	2	1	$A^T \otimes B^T \otimes C$				0.4596 (1)
44	5	2	2	$A^T \otimes B^T$			6.3215 (1)	
45	5	3	1	$A^T \otimes B^T \otimes C$				0.1697 (1)
46	5	3	2	$A^T \otimes B^T$		-5.8531 (1/3)	-6.7255 (2/3)	
47	5	4	1	$A^T \otimes (B^T vs. B^{SD}) \otimes C$				1.7269 (1)

Indexes				Type of contrasts	Strata			
h	j	k	l		(1)	(2)	(3)	(4)
48	5	4	2	$A^T \otimes (B^T vs. B^{SD})$			6.3870 (1)	
49	5	5	1	$A^T \otimes C$				2.0070 (1)
50	5	5	2	A^T		12.8280 (1)		
51	6	1	1	$(A^T vs. A^{SD}) \otimes B^T \otimes C$				-1.9039 (1)
52	6	1	2	$(A^T vs. A^{SD}) \otimes B^T$		9.3694 (1/3)	6.2666 (2/3)	
53	6	2	1	$(A^T vs. A^{SD}) \otimes B^T \otimes C$				-0.0036 (1)
54	6	2	2	$(A^T vs. A^{SD}) \otimes B^T$			7.9903 (1)	
55	6	3	1	$(A^T vs. A^{SD}) \otimes B^T \otimes C$				2.1098 (1)
56	6	3	2	$(A^T vs. A^{SD}) \otimes B^T$		-2.3251 (1/3)	-5.2255 (2/3)	
57	6	4	1	$(A^T vs. A^{SD}) \otimes (B^T vs. B^{SD}) \otimes C$				0.0606 (1)
58	6	4	2	$(A^T vs. A^{SD}) \otimes (B^T vs. B^{SD})$			-10.2104 (1)	
59	6	5	1	$(A^T vs. A^{SD}) \otimes C$				-2.7161 (1)
60	6	5	2	$A^T vs. A^{SD}$		-21.1928 (1)		
61	7	1	1	$B^T \otimes C$				-2.7653 (1)
62	7	1	2	B^T	-31.2485 (1/3)		-27.6021 (2/3)	
63	7	2	1	$B^T \otimes C$				-2.7085 (1)
64	7	2	2	B^T			-14.5547 (1)	
65	7	3	1	$B^T \otimes C$				0.3757 (1)
66	7	3	2	B^T	-2.0875 (1/3)		-4.1751 (2/3)	
67	7	4	1	$(B^T vs. B^{SD}) \otimes C$				7.1964 (1)
68	7	4	2	$B^T vs. B^{SD}$			38.3672 (1)	
69	7	5	1	C				45.098 (1)

(1) - the inter-block stratum, (2) - the inter-whole plot stratum, (3) - the inter-subplot stratum, (4) - the inter-sub-subplot stratum.

From Table 6 it can be seen that only contrasts among main effects of the test *A* treatments (A^T), the test *B* treatments (B^T) and the contrasts of the interaction effects of type $A^T \times B^T$, (A^T *vs.* A^{SD})$\times B^T$ are estimated with a different precision (two or three classes of efficiency). The contrasts with the first group of efficiency are estimated with full efficiency (= 1) in appropriate strata and the contrasts with the second group (not full efficiency) are estimated in two or three strata. The remaining contrasts are estimated as in a complete (orthogonal) SSP design with full efficiency in the sub-subplot stratum (c.f. Ambroży and Mejza, 2013). It means that their BLUEs under submodel in this stratum are BLUEs in the overall model of observations. In the statistical inference about those contrasts which are estimable in two or three strata we can use information about them separately from one stratum only. Advantage of used incomplete SSP design in the experiment is that more information about these contrasts is included in appropriate for them stratum. So we can choose this stratum to test the hypothesis connected with these contrasts.

Table 7 presents the analysis of variance in the strata (see, Section 2.4). One may notice, for instance, that in the fourth stratum the hypothesis $\mathrm{H}_{04}: \boldsymbol{\tau}'\mathbf{A}_4\boldsymbol{\tau} = 0$ refers to all basic contrasts which are estimable in the sub-subplot stratum (4). They are of types *C*, *A* × *C*, *B* × *C* and *A* × *B* × *C* and constitute so called “Treatments” in the stratum (4) as in Table 1.

From Table 7 we can calculate Mean Square for “Treatments” $\mathrm{MST}_4 = \frac{\mathrm{SST}_f}{\nu_{T4}} = \frac{2199.3641}{35} = 62.839$. Then, knowing the estimate of the error variance in the fourth stratum $\hat{\gamma}_4 = \mathrm{MSE}_4 = 0.6716$ we obtain that hypothesis H_{04} is rejected at α = 0.01 because F value = 93.566 is significant with empirical value *p* = 0.0000. More interesting are general hypotheses connected with the factor *C* and all interactions connected with it in the stratum (4). For instance the general hypothesis for interaction $A \times C$ can be written as $\mathrm{H}_0^{\mathrm{A\times C}}: \mathbf{W}'_{A\times C}\boldsymbol{\tau} = \mathbf{0}$, where $\mathbf{W}_{A\times C} = [\mathbf{c}_9\ \mathbf{c}_{19}\ \mathbf{c}_{29}\ \mathbf{c}_{39}\ \mathbf{c}_{49}\ \mathbf{c}_{59}]$, $\mathrm{r}(\mathbf{W}_{A\times C}) = 6$. This hypothesis is rejected at α = 0.01 because F value = 6,4942 is significant with empirical value *p* = 0.0000. We can say, that interaction between winter wheat genotype effects and the growth regulator effects is highly significant.

Then the general hypothesis connected with the factor *A* ($\mathrm{H}_0^A: \mathbf{W}'_A\boldsymbol{\tau} = \mathbf{0}$, $\mathrm{r}(\mathbf{W}_A) = 6$) is rejected at the level 0.01 (*p* = 0.0000) in the inter-whole plot stratum. As the result, at least two true average yields of the genotypes of winter wheat are not the same.

The general hypothesis connected with the factor B ($\mathrm{H}_0^B : \mathbf{W}_B'\boldsymbol{\tau} = \mathbf{0}$, $\mathrm{r}(\mathbf{W}_B) = 4$) is rejected at the level α = 0.01 (p = 0.0000) in the inter-subplot stratum. As the result, at least two true average yields at the considered nitrogen fertilization are different.

The general hypothesis connected with the main effects of the factor C is rejected at the level α = 0.01 (p = 0.0000) in the inter-sub-subplot stratum. As a result, the difference between true average yields of the chemical preparation – growth regulator are highly significant.

Similarly, the general hypothesis connected with the interaction effects between the factors A and B is rejected at the level α = 0.01 in the inter-subplots stratum. The remaining general hypotheses connected with interaction effects are testable in the fourth stratum only.

The next step, for example in the inter-sub-subplot stratum analysis (4), is to investigate the basic contrasts estimable in this stratum more closely (Table 8).

After the rejection of the general hypotheses in the strata, detailed study of estimable basic contrasts in these strata is recommended. We will illustrate this idea in the stratum (4) - between sub-subplots only. For instance all interaction contrasts of $A \times C$ type effects are estimable in this stratum. Five contrasts (with subscripts 9, 19, 29, 39, 49) relate to compare the interaction effects of the test genotypes and growth regulator combinations. While the interaction contrast $\mathbf{c}_{59}'\boldsymbol{\tau}$ concerns the comparison of the effect of standard treatment of factor A (standard variety) and the average effect of other treatments of this factor (test A treatments) under the chemical preparation - the growth regulator.

For example, it can be seen from Table 8, that through the hypothesis $\mathrm{H}_{04}^* : \mathbf{c}_9'\boldsymbol{\tau} = 0$ we can check if there is a significant difference between the true average yield of the test genotypes number 1, 2, 3, and the true average yield of other test varieties within growth regulator. We reject this hypothesis at the level α = 0.01. It means that the difference between above mentioned real averages of yield is significant.

Furthermore, we fail to reject the hypothesis $\mathrm{H}_{04}^* : \mathbf{c}_{19}'\boldsymbol{\tau} = 0$ declaring that there is no significant difference between the true average yield of the test varieties of the numbers 1, 2, 6 and the true average yield of other test varieties within growth regulator.

In turn, we reject the hypothesis $\mathrm{H}_{04}^* : \mathbf{c}_{59}'\boldsymbol{\tau} = 0$ at the significance level of 0.01, which represents a significant difference between the true averages of yield for some combinations $A \times C$. Probably the particular hypotheses for the contrasts with the subscripts 9, 29, 49, 59 are responsible for the rejection of the general hypothesis concerning the interaction effects of the type $A \times C$ in the (4) stratum.

Table 7. ANOVA for the incomplete SSP design considered

Sources of variation	DF	SS	MS	F	*p*
Stratum (1) – *inter-block analysis*					
Faktor *A*	2	9.0234	4.5117	3.2945	0.175
Faktor *B*	2	326.9426	163.4713	119.3705**	0.0014
$A \times B$	4	10.9767	2.7442	2.0039	0.2973
Error (1)	3	4.1083	1.3694		
Total (1) - *Bloks*	11	351.0510			
Stratum (2) – *inter-whole plot analysis*					
Faktor *A*	6	789.4661	131.5777	134.0705**	0.0000
$A \times B$	12	171.0212	14.2518	14.5218**	0.0000
Error (2)	30	29.4422	0.9814		
Total (2) – *Whole plots*	48	989.9295			
Stratum (3) – *inter-subplot analysis*					
Faktor *B*	4	2203.4188	550.8547	344.6495**	0.0000
$A \times B$	24	776.9855	32.3744	20.2555**	0.0000
Error (3)	92	147.0440	1.5983		
Total (3) - Subplots	120	3127.4483			
Stratum (4) – *inter-sub-subplot analysis*					
Faktor *C*	1	2033.8563	2033.8563	3028.3130**	0.0000
$A \times C$	6	26.1696	4.3616	6.4942**	0.0000
$B \times C$	4	66.9124	16.7281	24.9073**	0.0000
$A \times B \times C$	24	72.4258	3.0177	4.4933**	0.0000
Error (4)	145	97.3840	0.6716		
Total (4) - Sub-subplots	180	2296.7481			
Total	359	6765.1769			

** $p < 0.01$

Table 8. Detailed analysis on basic contrasts of type $A \times C$ in the inter-sub-subplot stratum

Stratum (4) - Analysis of sub-subplots					
Source	**DF**	**SS**	**MS**	**F**	***p***
Contrasts of type $A \times C$ including:	6	26.1696	4.3616	6.4942**	0.0000
$\mathbf{c}_9'\tau$	1	9.0845	9.0845	13.5267**	0.0003
$\mathbf{c}_{19}'\tau$	1	0.1064	0.1064	0.1584	0.6912
$\mathbf{c}_{29}'\tau$	1	4.8864	4.8864	7.2758**	0.0078
$\mathbf{c}_{39}'\boldsymbol{\tau}$	1	0.6868	0.6868	1.0226	0.3136
$\mathbf{c}_{49}'\boldsymbol{\tau}$	1	4.0281	4.0281	5.9978*	0.0155
$\mathbf{c}_{59}'\boldsymbol{\tau}$ a	1	7.3774	7.3774	10.9848**	0.0012
Rest	29	2173.1945	74.9377		
Error (4)	145	97.3840	0.6716		
Total (4) - sub-subplots	180	2296.7481			

** $p < 0.01$; * $p < 0.05$

For a complete discussion on study of the significance of contrasts we will show how to use the basic treatment contrasts to estimate any treatment contrasts (see, Section 2.3). They include, inter alia, any comparisons of each test variety effect with the standard variety effect within growth regulator, i.e. between interaction effects for $A_1 - A_7$, $A_2 - A_7$, $A_3 - A_7$, $A_4 - A_7$, $A_5 - A_7$ and $A_6 - A_7$ within the factor *C*. Let us denote the mentioned contrasts by $\mathbf{s}_1'\boldsymbol{\tau}$, $\mathbf{s}_2'\boldsymbol{\tau}$, $\mathbf{s}_3'\boldsymbol{\tau}$, $\mathbf{s}_4'\boldsymbol{\tau}$, $\mathbf{s}_5'\boldsymbol{\tau}$ and $\mathbf{s}_6'\boldsymbol{\tau}$, respectively. In Table 9 there are forms of them, their estimates, variance estimates of them and calculated values of test statistic F and corresponding *p* values used to test particular hypothesis of the form $\mathrm{H}_{04}^{*}: \mathbf{s}'\boldsymbol{\tau} = 0$.

Consider now any contrast $\mathbf{s}_1'\boldsymbol{\tau}$ for instance between the test variety A_1 effect and standard variety (A_7) effect within growth regulator. It can be written (for the treatment combination form) as

$$\mathbf{s}_1'\boldsymbol{\tau} = [1, 0, 0, 0, 0, 0, -1] \otimes [1, 1, 1, 1, 1] \otimes [1, -1]\boldsymbol{\tau}.$$

Table 9. Set of the elementary contrasts for the test varieties vs. standard within the growth regulator

Basic contrasts	Coefficients λ_{4h} for the elementary contrasts					
j	$\mathbf{s}_1'\boldsymbol{\tau}$	$\mathbf{s}_2'\boldsymbol{\tau}$	$\mathbf{s}_3'\boldsymbol{\tau}$	$\mathbf{s}_4'\boldsymbol{\tau}$	$\mathbf{s}_5'\boldsymbol{\tau}$	$\mathbf{s}_6'\boldsymbol{\tau}$
9	$5/6\sqrt{2}$	$5/6\sqrt{2}$	$5/6\sqrt{2}$	$-5/6\sqrt{2}$	$-5/6\sqrt{2}$	$-5/6\sqrt{2}$
19	$5/12$	$5/12$	$-5/6$	$-5/12$	$-5/12$	$5/6$
29	$5/4\sqrt{3}$	$-5/4\sqrt{3}$	0	$5/4\sqrt{3}$	$-5/4\sqrt{3}$	0
39	$5/12$	$5/12$	$-5/6$	$5/12$	$5/12$	$-5/6$
49	$5/4\sqrt{3}$	$-5/4\sqrt{3}$	0	$-5/4\sqrt{3}$	$5/4\sqrt{3}$	0
59	$25/6\sqrt{10}$	$25/6\sqrt{10}$	$25/6\sqrt{10}$	$25/6\sqrt{10}$	$25/6\sqrt{10}$	$25/6\sqrt{10}$
$\widehat{(\mathbf{s}'\boldsymbol{\tau})}_4$	0.7597	-5.3277	-0.8403	-5.4173	-5.7110	-4.9360
Estimated variances	2.3319	2.3319	2.3319	2.3319	2.3319	2.3319
F	0.2475	12.1722**	0.3028	12.5851**	13.9867**	10.4483**
p	0.6196	0.0006	0.583	0.0005	0.0005	0.0015

** $p < 0.01$

It can be shown (see, Section 2.2) that this contrast is estimable in the sub-subplot stratum (4). From Table 6 it is easily seen that in this stratum 35 basic contrasts $\mathbf{c}_h'\boldsymbol{\tau}$, h = 1, 3, 5, 7,…, 69 are estimable. It means that any treatment contrast estimable in the stratum (4) can be expressed by them. Because of the $\mathbf{r}^{-\delta}$- orthonormality of the vectors $\mathbf{c}_j$, j = 1, 2,..., h, we may limit this set to 6 vectors. So we obtain

$$\mathbf{s}_1'\boldsymbol{\tau} = \frac{5}{6\sqrt{2}}\mathbf{c}_9'\boldsymbol{\tau} + \frac{5}{12}\mathbf{c}_{19}'\boldsymbol{\tau} + \frac{5}{4\sqrt{3}}\mathbf{c}_{29}'\boldsymbol{\tau} + \frac{5}{12}\mathbf{c}_{39}'\boldsymbol{\tau} + \frac{5}{4\sqrt{3}}\mathbf{c}_{49}'\boldsymbol{\tau} + \frac{25}{6\sqrt{10}}\mathbf{c}_{59}'\boldsymbol{\tau}\ .$$

It is easy to check the contrast is estimated with full efficiency ($\mathrm{E}_4(\mathbf{s}_1'\boldsymbol{\tau}) = 1$) in the inter-sub-subplot stratum. Taking into account the estimates $\widehat{(\mathbf{c}_h'\boldsymbol{\tau})}_4$, h = 9, 19, 29, 39, 49, 59 (see, Table 6) the estimate of the contrast $\mathbf{s}_1'\boldsymbol{\tau}$ in the stratum (4) is equal to

$$(\widehat{\mathbf{s}_1'\boldsymbol{\tau}})_4 = \frac{5}{6\sqrt{2}}(3.014) + \frac{5}{12}(-0.3262) + \frac{5}{4\sqrt{3}}(2.2105) + \frac{5}{12}(-0,8287) + \frac{5}{4\sqrt{3}}(2.007) + \frac{25}{6\sqrt{10}}(-2.7161) = 0.7597.$$

From Table 7 we have that variance estimate of the error in the stratum (4) is equal to $\hat{\gamma}_4 = \mathrm{MSE}_4 = 0.6716$. So the estimate of the variance of the contrast estimate is 2.3319, then F value = [$(\widehat{\mathbf{s}_1'\boldsymbol{\tau}})_4$]^2 / 2.3319 = 0.2475 <1.

It means that we fail to reject the hypothesis $\mathrm{H}_{04}^{*} : \mathbf{s}_1'\boldsymbol{\tau} = 0$ declaring that there is no significant difference between the true average yield of the test variety number 1 and the true average yield of the standard variety within the growth regulator.

Moreover, using information from Table 9 we can say conclusions connected with testing other hypotheses for any contrasts. The hypotheses associated with the contrasts $\mathbf{s}_2'\boldsymbol{\tau}$, $\mathbf{s}_4'\boldsymbol{\tau}$, $\mathbf{s}_5'\boldsymbol{\tau}$ and $\mathbf{s}_6'\boldsymbol{\tau}$ should be rejected only ($p < 0.01$).

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