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BALANCE AND EFFICIENCY OF SOME AUGMENTED SPLIT-BLOCK-PLOT DESIGN

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Summary

A construction procedure of an augmented split-block-plot design with control subplot treatments is presented. In the modeling data the structure of an experimental material and a four-step randomization scheme are taken into account. With respect to the analysis of the obtained randomization model with six strata the approach typical to the multistratum experiments with orthogonal block structure is adapted. A numerical example is presented to illustrate the method of the construction and statistical properties of the final design.

Key words and phrases: augmented block design, control treatments, general balance, split-block-plot design, stratum efficiency, test treatments

Classification AMS 2010: 62K10, 62K15

1. Introduction

Some experimental designs used in an agricultural research for three-or-more factor experiments are extensions of either a split-plot design or a split-block design (cf. Gomez and Gomez, 1984). The split-block-plot (SBP) design is the extension of the split-block design in which the intersection plot is divided into subplots to accommodate a third factor. Another term of the design is the strip-split-plot design (cf. Gomez and Gomez, 1984) or split-block-split-plot design (cf. Trętowski and Wójcik, 1988). In field and glasshouse trials the complete SBP designs are commonly used in practice, then all levels of the factors are orthogonal to blocks. The complete or incomplete SBP designs were considered by Ambroży and Mejza (2002a, 2002b, 2006). In the incomplete designs not all treatment combinations are found within the blocks (for example when an experimental material for certain treatments is limited). Then some contrasts among effects of the treatment combinations can be estimated with not full efficiency in appropriate strata of the model of observations.

In the paper we consider a situation when the SBP design is augmented by a new group of subplot treatments (called control subplot treatments) which are to be replicated more than the test subplot treatments. Usually, if we are interested, among other things, in the comparison of the basic (test) group of the treatments with the additional (control) group of the treatments with full efficiency, we could set up an experiment in an augmented design. Augmented designs can be generated by designs from a class of *augmented block designs* known from the literature also as s*upplemented block designs*, introduced for one-factor experiments (cf. Pearce 1960, Federer 1961, Corsten 1962, Caliński 1971, Caliński and Ceranka 1974, Singh and Dey 1979, Puri et al. 1977, Kachlicka and Mejza 2000a, 2000b).

In the paper we present a randomization model, statistical properties and their consequences for an analysis of some three factor experiment set up in an augmented by control subplot SBP design.

2. Assumptions and notations

Consider an $(s \times t \times w)$- factorial experiment in which the first factor, say A, has s levels A_1, A_2, …, A_s, the second factor, say B, has t levels B_1, B_2, …, B_t and the third factor, say C, has w levels C_1, C_2, …, C_w. Let v (= stw) be the number of all treatment combinations.

We assume an experimental material consists of b blocks (*b is not to be prime number*) which can be grouped into R superblocks of the same size. So, each superblock contains b/R blocks. *It should be underlined that grouping of blocks in the superblocks is strictly connected with an applied here constructing method of a final design.* It is assumed that each superblock contains blocks in which those same treatment combinations occur unlike in some cases than other superblocks, see section 3.

Generally, in SBP designs each block has a row-column structure with k_1 rows and k_2 columns. Then, each intersection plot (called a whole plot) is divided into k_3 subplots. The rows correspond to the levels of factor A (row treatments), the columns correspond to the levels of factor B (column treatments), and the subplots are to accommodate the levels of factor C (subplot treatments or C treatments).

In the paper it is assumed that SBP design is incomplete with respect to the levels of the C factor while other factors, A and B, are treated as in a complete SBP design or $k_1 = s$, $k_2 = t$, $k_3 < w$.

The considered model of observations has a form and properties strictly connected with performed randomization processes in the experiment. The randomization scheme of the SBP design consists of four randomization steps performed independently, that is by randomly permuting blocks, rows, columns and subplots (within each whole plot). As a result the mixed model is marked by the following form (cf. Ambroży and Mejza 2002a, 2002b, 2006):

$$\mathrm{E}(\mathbf{y}) = \boldsymbol{\Delta}'\boldsymbol{\tau}, \quad \mathrm{Cov}(\mathbf{y}) = \mathbf{V}(\gamma), \tag{2.1}$$

where $\boldsymbol{\Delta}'$ is a known design matrix for v treatment combinations, and $\boldsymbol{\tau}$ ($v\times 1$) is the vector of fixed treatment combination effects. According to the orthogonal block structure of the SBP designs, the dispersion matrix $\mathbf{V}(\gamma)$ can be expressed by $\mathbf{V}(\gamma) = \sum_{f=0}^{5}\gamma_f \mathbf{P}_f$, where $\gamma_f \geq 0$ and $\{\mathbf{P}_f\}$ are a family of known pairwise orthogonal projectors adding up to the identity matrix (cf. Houtman and Speed, 1983). The range space $\Re\{\mathbf{P}_f\}$ of $\mathbf{P}_f$, f = 0, 1,.., 5, is termed the f-th stratum of the model and $\{\gamma_f\}$ are unknown stratum variances. This model will be analyzed using the methods developed for multistratum experiments (Nelder, 1965). So, we have zero stratum (0) generated by the vector of ones, inter-block stratum (1), inter-row (within the block) stratum (2), inter-column (within the block) stratum (3), inter-whole plot (within the block) stratum (4) and inter-subplot (within the whole plot) stratum (5).

It is well known that statistical properties of the considered SBP design are strictly connected with the algebraic properties of the stratum information matrices for the treatment combinations. Generally these matrices have the following forms:

$$\mathbf{A}_f = \boldsymbol{\Delta}\mathbf{P}_f\boldsymbol{\Delta}', \quad f = 1, 2, \ldots, 5\,. \tag{2.2}$$

In the considered case of the SBP design you can find the matrices (2.2) in Ambroży and Mejza (2002a, 2002b, 2006). So, eingenvalues of these information matrices are called stratum efficiency factors and corresponding to them eigenvectors generate orthogonal (basic) contrasts among effects of the treatment combinations. The algebraic structures of the matrices (2.2) imply information about stratum efficiency of the augmented SBP design with respect to particular basic contrasts.

3. Construction method of the augmented SBP design

In the paper we present a construction of the incomplete SBP design with respect to the levels of the factor C augmented within whole plots by the control C treatments. The remaining factors (A treatments and B treatments) are arranged as in a complete SBP design.

In the method of the construction described below we use an augmented block design $d^* = \begin{bmatrix} \tilde{d}_1 \\ \tilde{d}_2 \end{bmatrix}$ as a generating design for w subplot treatments (C treatments). We assume that subplot treatments consist of two groups: $w = w_1 + w_2$, where w_1 test (basic) C treatments are allocated in the $\tilde{d}_1$ subdesign which is an incomplete block design and w_2 additional (control) C treatments – in the $\tilde{d}_2$ subdesign represented by a randomized complete block (RCB) design. So, the incidence matrix $\mathbf{N}_{d*}$ has the following form (cf. Kachlicka and Mejza, 2000b):

$$\mathbf{N}_{d*} = \begin{bmatrix} \tilde{\mathbf{N}}_1 \\ \tilde{\mathbf{N}}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{I}_{\tilde{R}} \otimes \mathbf{1}_{\tilde{k}_1} \mathbf{1}'_{\tilde{b}_1/\tilde{R}} \\ \mathbf{1}_{w_2} \mathbf{1}'_{\tilde{b}_1} \end{bmatrix}. \tag{3.1}$$

From (3.1) follows the $\tilde{d}_1$ subdesign has $\tilde{b}_1$ blocks with $\tilde{k}_1$ ($< w_1$) units and the $\tilde{d}_2$ subdesign has $\tilde{b}_2$ (= $\tilde{b}_1$) blocks with $\tilde{k}_2$ (= w_2) units. It is assumed that $\tilde{b}_1$ is not to be prime number and so the blocks can be grouped into $\tilde{R}$ superblocks of the same size ($\tilde{b}_1 / \tilde{R}$ blocks). The superblocks differ in the test C treatments only. So, we have $w_1 = \tilde{R}\tilde{k}_1$ test C treatments. The blocks inside each superblock of the $\tilde{d}_1$ subdesign are then supplemented by $\tilde{k}_2$ (= w_2) units to accomodate w_2 additional (control) C treatments. Therefore, the generating design d^* has $b^* = \tilde{b}_1$ blocks and the number of units inside each block is equal to $k^* = \tilde{k}_1 + w_2$. These parameters are taken into account in the constructing method of the final (SBP) design. We will investigate also algebraic properties of the d^* design being useful to construct this SBP design.

Let $\mathbf{C}_{d*}$ $(= \mathbf{C}_C)$ be information matrix for C treatments in the augmented design d^* with the following positive eigenvalues: $\varepsilon_0^* = 1$ and $\varepsilon_1^* = \frac{w_2}{k_3}$ and their multiplicities $\rho_0^* = 1 + R(\tilde{k}_3 - 1) + (w_2 - 1) = w_1 + w_2 - R$ and $\rho_1^* = R - 1$, respectively. It can be shown that

- the first class of the efficiency equal to ε_0^* (= 1) is connected with the comparison: 1) between the basic (test) group and the additional (control) group of the C treatments (C^T *vs.* C^C), 2) among the basic (test) C treatments inside each superblock (C_1^T), 3) among the additional (control) C treatments (C^C),
- the second class of efficiency equal to ε_1^* refers to the comparisons among the basic (test) C treatments between the superblocks (C_2^T).

In the present paper the construction method for three factor experiments is based on Kronecker product of matrices denoted by ⊗ (cf. Ambroży and Mejza, 2006). Let $\mathbf{N}_1$ be the treatment combinations vs. blocks incidence matrix of the augmented SBP design. Then we have:

$$\mathbf{N}_1 = \mathbf{1}_s \otimes \mathbf{1}_t \otimes \mathbf{N}_{d*}, \tag{3.2}$$

where $\mathbf{N}_{d*}$ is given in (3.1) and $\mathbf{1}_s$ and $\mathbf{1}_t$ denote one block $(b_1 = b_2 = 1)$ incidence matrices for factors A and B, respectively. Generally we can write parameters of the three factor design as follows: $v = stw$, $b = b_1 b_2 b_3$, $k = k_1 k_2 k_3$, $\mathbf{r} = \mathbf{r}_A \otimes \mathbf{r}_B \otimes \mathbf{r}_C$, where *v*, *b*, *k*, **r**, $\mathbf{r}_A$, $\mathbf{r}_B$, $\mathbf{r}_C$ denote the number of the treatment combinations, the number of the blocks, the size of the blocks, the vector of replicates of the treatment combinations, the vector of replicates of A treatments , B treatments and C treatments, respectively. In our case of the SBP design the number of blocks and the size of the blocks are generated by the augmented block design for the C treatments only, i.e. $b_3 = b^* = \tilde{b}_1$ and $k_3 = k^* = \tilde{k}_1 + w_2$. Finally we express the parameters of the considered SBP design in the following form:

$$v = stw,\ b = b_3,\ R = \tilde{R},\ k = stk_3,\ n = bk,$$
$$\mathbf{r} = \mathbf{1}_s \otimes \mathbf{1}_t \otimes \mathbf{r}_C,\ \mathbf{r}_C = [(\tilde{b}_1 / \tilde{R})\mathbf{1}'_{w_1} \vdots \tilde{b}_1\mathbf{1}'_{w_2}]', \tag{3.3}$$

It is convenient to introduce abbreviations to describe the properties such as efficiency and balance of the augmented SPB design. Let $M_f\{q, \alpha\}$ denote the property that q contrasts among treatments of factor M (or interaction contrasts) are estimated with efficiency α in the f-th stratum. In other words, we say that the design is $M_f\{q, \alpha\}$ - balanced. Particularly, for α = 1, the design is $M_f\{q, 1\}$ - orthogonal.

Following algebraic properties of the stratum information matrices for the treatment combinations (2.2) and the information matrix for C treatments $\mathbf{C}_{d*}$ we have:

Corollary 3.1. The augmented SBP design with the incidence matrix definite in (3.2) is:
$(C_2^T)_1\{\rho_1^*, 1-\varepsilon_1^*\}$– balanced,
$A_2\{s-1, 1\}$– orthogonal, $(A \times C_2^T)_2\{(s-1)\rho_1^*, 1-\varepsilon_1^*\}$– balanced,
$B_3\{t-1, 1\}$– orthogonal, $(B \times C_2^T)_3\{(t-1)\rho_1^*, 1-\varepsilon_1^*\}$– balanced,
$(A \times B)_4\{(s-1)(t-1), 1\}$– orthogonal, $(A \times B \times C_2^T)_4\{(s-1)(t-1)\rho_1^*, 1-\varepsilon_1^*\}$ – balanced,
$(C_2^T)_5\ \{\rho_1^*,\ \varepsilon_1^*\}$ – balanced, $(A \times C_2^T)_5\ \{(s-1)\rho_1^*,\ \varepsilon_1^*\}$ – balanced, $(B \times C_2^T)_5\ \{(t-1)\rho_1^*,\ \varepsilon_1^*\}$ – balanced, $(A \times B \times C_2^T)_5\{(s-1)(t-1)\rho_1^*,\ \varepsilon_1^*\}$– balanced,
$(C_1^T)_5\{(w_1-R), 1\}$– orthogonal, $(A \times C_1^T)_5\{(s-1)(w_1-R), 1\}$– orthogonal, $(B \times C_1^T)_5\{(t-1)(w_1-R), 1\}$– orthogonal, $(A \times B \times C_1^T)_5\{(s-1)(t-1)(w_1-R), 1\}$– orthogonal,

$(C^C)_5\{(w_2-1),\ 1\}$– orthogonal, $(A\times C^C)_5\{(s-1)(w_2-1),\ 1\}$– orthogonal, $(B\times C^C)_5\{(t-1)\ (w_2-1),\ 1\}$– orthogonal, $(A\times B\times C^C)_5\{(s-1)(t-1)(w_2-1),\ 1\}$– orthogonal,
$(C^T\ vs.\ C^C)_5\{(1,\ 1\}$– orthogonal, $(A\times(C^T\ vs.\ C^C))_5\{(s-1),\ 1\}$– orthogonal, $(B\times(C^T\ vs.\ C^C))_5\{(t-1),\ 1\}$– orthogonal, $(A\times B\times(C^T\ vs.\ C^C))_5\{(s-1)(t-1),\ 1\}$– orthogonal.

4. Example

To illustrate our considerations we will characterize the estimation of the orthogonal contrasts in a certain $(2\times 2\times 7)$- factorial experiment. Assume that experiment is set up in the augmented SBP design in which the A treatments ($s=2$) and B treatments ($t=2$) are arranged as in a complete SBP design whereas the C treatments ($w=7$) are allocated on the subplots according to the incidence matrix as follows:

$$\mathbf{N}_{d^*}=\begin{bmatrix}1&1&0&0\\1&1&0&0\\0&0&1&1\\0&0&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{bmatrix}. \tag{4.1}$$

From (4.1) we can notice, four test (basic) C treatments ($w_1=4$) are allocated in an incomplete subdesign in four blocks ($\tilde{b}_1=4$) of size two ($\tilde{k}_1=2$). These blocks can be grouped into $\tilde{R}=2$ superblocks. Each superblock of the basic design is supplemented by three additional C treatments ($w_2=3$). The parameters of the augmented design d^* for the C treatments are following:

Table 1. Stratum efficiency factors of the augmented SBP design

Types of contrasts	**Df**	**Strata**				
		1	**2**	**3**	**4**	**5**
A	1		1			
B	1			1		
C_1^T	2					1
C_2^T	1	0.4				0.6
C^C	2					1
C^T *vs* C^C	1					1
$A \times B$	1				1	
$A \times C_1^T$	2					1
$A \times C_2^T$	1		0.4			0.6
$A \times C^C$	2					1
$A \times (C^T$ *vs* $C^C)$	1					1
$B \times C_1^T$	2					1
$B \times C_2^T$	1			0.4		0.6
$B \times C^C$	2					1
$B \times (C^T$ *vs* $C^C)$	1					1
$A \times B \times C_1^T$	2					1
$A \times B \times C_2^T$	1				0.4	0.6
$A \times B \times C^C$	2					1
$A \times B \times (C^T$ *vs* $C^C)$	1					1

Df (*degrees of freedom*) – numbers of the particular types of the contrasts estimable in the strata;

1 – the inter-block stratum, 2 – the inter-row stratum, 3 – the inter-column stratum,
4 – the inter-whole plot stratum, 5 – the inter-subplot stratum

$$w = w_1 + w_2 = 7,\ b^* = 4,\ k^* = 5,\ \mathbf{r}_C = [2, 2, 2, 2, 4, 4, 4]',$$

$$\varepsilon_0^* = 1,\ \rho_0^* = 5,\ \varepsilon_1^* = 0{,}6,\ \rho_1^* = 1.$$

So, the parameters (3.3) of the final design are equal to

$$v = 28,\quad b = 4,\quad k = 20,\quad n = 80,\quad \mathbf{r} = \mathbf{1}_2 \otimes \mathbf{1}_2 \otimes [2, 2, 2, 2, 4, 4, 4]'.$$

In the table 1 we express the efficiency factors of the considered SBP design generated by the augmented block design d^* for those orthogonal contrasts which are estimable in suitable for them strata (see Corollary 3.1.).

It is worth noticing that in the presented augmented SBP design all contrasts among C treatments are estimated in the inter-subplot stratum (5). We loss information about the contrasts among the test C treatments (C_2^T) and interaction contrasts connected with them, only. These contrasts are estimated with not full efficiency in the strata (1) or (2) or (3) or (4) and in the inter-subplot stratum (5) depending on the type of the contrast. The remaining contrasts are estimable with full efficiency in appropriate for them strata as in a complete SBP design. In other words, the considered design is: $(C_2^T)_1\{1;\ 0.4\}$-balanced, $(C_2^T)_5\{1,\ 0.6\}$- balanced, $(A \times C_2^T)_2\{1;\ 0.4\}$- balanced, $(A \times C_2^T)_5\{1,\ 0.6\}$- balanced, $(B \times C_2^T)_3\{1;\ 0.4\}$- balanced, $(B \times C_2^T)_5\{1;\ 0.6\}$- balanced, $(A \times B \times C_2^T)_4\{1;\ 0.4\}$- balanced and $(A \times B \times C_2^T)_5\{1;\ 0.6\}$- balanced. For the remaining contrasts the augmented by subplot treatments SBP design is orthogonal in appropriate strata.

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ZRÓWNOWAŻENIE I EFEKTYWNOŚĆ PEWNEGO ROZSZERZONEGO UKŁADU SPLIT-BLOCK-PLOT

Streszczenie

W pracy przedstawiono metodę konstrukcji układu niekompletnego split-block-plot rozszerzonego przez obiekty kontrolne na poletkach małych w obrębie poletek dużych. W modelowaniu danych brane są pod uwagę struktura materiału doświadczalnego i czterostopniowy schemat randomizacyjny. Do analizy uzyskanego w ten sposób randomizacyjnego modelu z sześcioma warstwami zastosowano metodę właściwą dla doświadczeń wielowarstwowych z ortogonalną strukturą blokową. Zaprezentowano numeryczny przykład ilustrujący metodę konstrukcji i właściwości statystyczne uzyskanego układu.

Słowa kluczowe: rozszerzony układ blokowy, obiekty kontrolne, ogólne zrównoważenie, układ split-block-plot, warstwowa efektywność, obiekty testowe

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