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The growth curve model with heterogeneous fractional decreasing correlation structure

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Summary

In this text we will derive estimators of unknown parameters for growth curve model with different decrease of dependency in groups and we will compare them for the case of two groups. We will use method of maximum likelihood and method of unbiased estimating equations. This article is a continuation of article (Rusnačko and Žežula 2015) in which the used special correlation structure was introduced.

Key words: growth curve model, uniform correlation structure, serial correlation structure, heterogeneous fractional decreasing correlation structure, estimators of variance parameters, unbiased estimating equations, maximum likelihood estimators

AMS classification: 15A63, 62H12, 62P99

1 Introduction

The growth curve model (GCM) represents connection between regression analysis and analysis of variance and it has the following form

$$Y = XBZ + \varepsilon, \quad \mathrm{E}(\varepsilon) = 0, \quad \mathrm{Var}(\mathrm{vec}\ \varepsilon) = \Sigma \otimes I, \tag{1.1}$$

where $Y_{n\times p}$ is matrix of n $p-$dimensional observations, $X_{n\times m}$ is ANOVA matrix, $B_{m\times r}$ is matrix of unknown parameters, $Z_{r\times p}$ is matrix of regression constants, $\varepsilon_{n\times p}$ is matrix of random errors which has normal distribution, $I_{n\times n}$ is identity matrix, $\Sigma_{p\times p}$ is variance matrix of rows of matrix Y (variance matrix of a single observation) and vec operator transforms a matrix into a vector by stacking the columns of the matrix one underneath the other. This model was introduced by Potthoff and Roy already in 1964. The variance matrix contains a lot of unknown parameters, therefore it is useful to reduce their number by considering a simpler structure. The most commonly used structures are the uniform correlation structure and the serial correlation structure.

If the variance matrix has equal diagonal elements and equal off-diagonal elements we talk about a uniform correlation structure, which is of the form

$$\Sigma = \sigma^2 \left[(1-\varrho)\, I + \varrho \mathbf{1}\mathbf{1}'\right] = \sigma^2 \begin{pmatrix} 1 & \varrho & \dots & \varrho \\ \varrho & 1 & \dots & \varrho \\ \vdots & \vdots & \ddots & \vdots \\ \varrho & \varrho & \dots & 1 \end{pmatrix}, \tag{1.2}$$

where $\sigma^2 > 0$ and $\varrho \in \left(-\frac{1}{p-1}, 1\right)$ are unknown parameters. Problem of this model can be decreasing dependence among more remote observations in space (or time), which is not reflected in this structure.

Serial or first order autoregressive correlation structure is natural for time series and repeated measurements and is of the form

$$\Sigma = \sigma^2 \begin{pmatrix} 1 & \varrho & \dots & \varrho^{p-1} \\ \varrho & 1 & \dots & \varrho^{p-2} \\ \vdots & \vdots & \ddots & \vdots \\ \varrho^{p-1} & \varrho^{p-2} & \dots & 1 \end{pmatrix} = \sigma^2 \left(\varrho^{|i-j|}\right) = \sigma^2 \sum_{i=2}^{p} \varrho^{i-1} W_i + \sigma^2 I, \tag{1.3}$$

where $\sigma^2 > 0$ and $\varrho \in (-1, 1)$ are unknown parameters, $W_k = (w_{ij}(k))$ is $p \times p$ matrix whose (i, j) entry

$$w_{ij}(k) = \begin{cases} 1, & \text{if } |i-j| = k-1, \\ 0, & \text{if } |i-j| \neq k-1, \end{cases}$$

for $k = 2, \dots, p \geq 3$. But the dependence among observations may not be exponentially decreasing. Therefore it is useful to consider a model with slower decrease of dependency than first-order autoregressive process.

Let us consider a correlation structure of the form

$$\Sigma = \sigma^2 \left[(1-\varrho)I + \varrho A\right] = \sigma^2 \begin{pmatrix} 1 & \varrho a_1 & \varrho a_2 & \dots & \varrho a_{p-1} \\ \varrho a_1 & 1 & \varrho a_1 & \dots & \varrho a_{p-2} \\ \varrho a_2 & \varrho a_1 & 1 & \dots & \varrho a_{p-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \varrho a_{p-1} & \varrho a_{p-2} & \varrho a_{p-3} & \dots & 1 \end{pmatrix}, \tag{1.4}$$

where $\sigma^2 > 0$ and $\varrho \in (-1, 1)$ are unknown variance parameters and the matrix A is the known symmetric Toeplitz matrix with elements $(1, a_1, a_2, \ldots, a_{p-1})$. This structure can simulate a slower decrease of dependence than exponential. Therefore it can be viewed as a transition between the uniform and the serial correlation structure. Estimators of unknown parameters for this special correlation structure, their statistical properties and their comparison are derived in (Rusnačko and Žežula 2015).

In this article we extend these issues by consideration of different decrease of dependency in the groups of the GCM. In the following section we will introduce this problem and we will derive estimators of unknown parameters based on unbiased estimating equations and the method of maximum likelihood. We will show some simulations to compare them in the case of two groups.

2 The GCM with different decrease of dependency in groups

Let us consider growth curve model with t groups and different decrease of dependency in these groups. Formally, we have model (1.1) with

$$Y = \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_t \end{pmatrix}, \quad X = \mathrm{diag}\left(\mathbf{1}_{n_1}, \mathbf{1}_{n_2}, \ldots, \mathbf{1}_{n_t}\right), \quad B = (\beta_1, \beta_2, \ldots, \beta_t)', \quad \varepsilon = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_t \end{pmatrix},$$

where

$$\mathrm{E}\ (\varepsilon_i) = 0, \ \ \mathrm{Var}\ (\mathrm{vec}\ \varepsilon_i) = \Sigma_i \otimes I_{n_i}, \ \ \text{and} \ \ \Sigma_i = \sigma^2\left[(1-\varrho)I + \varrho A_i\right],$$

where Y_i and ε_i are $n_i \times p$ matrices, β_i is vector of unknown parameters corresponding to $i-$th group, matrices A_i are symmetric $p \times p$ Toeplitz matrices with elements $(1, a_{1,i}, a_{2,i}, \ldots, a_{p-1,i})$ and each variance matrix Σ_i is positive semi-definite, $i = 1, 2, \ldots, t$. Using the transposed model we get

$$\Lambda = \mathrm{var}\ (\mathrm{vec}\ \varepsilon') = \begin{pmatrix} I_{n_1} \otimes \Sigma_1 & 0 & \ldots & 0 \\ 0 & I_{n_2} \otimes \Sigma_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & I_{n_t} \otimes \Sigma_t \end{pmatrix}, \tag{2.1}$$

so for using the method of unbiased estimating equations we estimate each diagonal block using uniformly minimum variance unbiased invariant estimator of variance matrix which is for each group in the form

$$S_i = \frac{1}{n_i - 1} Y_i' M_{\mathbf{1}_{n_i}} Y_i = \frac{1}{n_i - 1} Y_i' \left(I_{n_i} - \frac{1}{n_i} \mathbf{1}_{n_i} \mathbf{1}_{n_i}' \right) Y_i, \quad i = 1, 2, \ldots, t. \tag{2.2}$$

In the next subsection we will derive estimators of unknown parameters based on unbiased estimating equations and estimators (2.2).

2.1 Estimators based on unbiased estimating equations

To use this method of estimating we need mean value of trace and mean value of sum of all elements of each estimator of variance matrix (2.2). For simplicity let us denote

$$n = \sum_{j=1}^{t} n_j \quad \text{and} \quad R_i = \sum_{j=1}^{p-1} j a_{p-j,i}, \quad i = 1, 2, \ldots, t.$$

Since estimators (2.2) are unbiased, then

$$\mathrm{E}\left(\mathrm{Tr}\left(S_i\right)\right) = p\sigma^2 \quad \text{and} \quad \mathrm{E}\left(\mathbf{1}' S_i \mathbf{1}\right) = \sigma^2 \left(p + 2\varrho R_i\right), \quad i = 1, 2, \ldots, t.$$

Thus, unbiased estimating equations are

$$\sum_{i=1}^{t} n_i \mathrm{Tr}(S_i) - np\widehat{\sigma}^2 = 0,$$

$$\sum_{i=1}^{t} n_i \mathbf{1}' S_i \mathbf{1} - \widehat{\sigma}^2 \sum_{i=1}^{t} n_i \left(p + 2\widehat{\varrho} R_i\right) = 0,$$

which implies that estimators of unknown parameters based on unbiased estimating equations are of the form

$$\widehat{\sigma}^2 = \frac{\sum_{i=1}^{t} n_i \mathrm{Tr}(S_i)}{np} \tag{2.3}$$

and

$$\widehat{\varrho} = \frac{np}{2\sum_{i=1}^{t} n_i R_i} \left[\frac{\sum_{i=1}^{t} n_i \mathbf{1}' S_i \mathbf{1}}{\sum_{i=1}^{t} n_i \mathrm{Tr}(S_i)} - 1 \right]. \tag{2.4}$$

To derive properties of these estimators we use the fact that for estimators (2.2) hold

$$\mathrm{Var}(\mathrm{vec}\ S_i) = \frac{1}{n_i - 1} (I_{p^2} + K_{pp})(\Sigma_i \otimes \Sigma_i),$$

which implies

$$\mathrm{Var}(\mathrm{Tr}\ (S_i)) = (\mathrm{vec} I_p)' \, \mathrm{Var}\left(\mathrm{vec} S_i\right) \left(\mathrm{vec} I_p\right) = \frac{2}{n_i - 1} \mathrm{Tr}(\Sigma_i^2),$$

$$\mathrm{Var}(\mathbf{1}' S_i \mathbf{1}) = (\mathrm{vec} J_p)' \, \mathrm{Var}\left(\mathrm{vec} S_i\right) \left(\mathrm{vec} J_p\right) = \frac{2}{n_i - 1} (\mathbf{1}' \Sigma_i \mathbf{1})^2,$$

$$\mathrm{Cov}(\mathbf{1}' S_i \mathbf{1}, \mathrm{Tr}\ (S_i)) = (\mathrm{vec} J_p)' \, \mathrm{Var}\left(\mathrm{vec} S_i\right) \left(\mathrm{vec} I_p\right) = \frac{2}{n_i - 1} \mathbf{1}' \Sigma_i^2 \mathbf{1}, \quad i = 1, 2, \ldots, t,$$

where $J_p = \mathbf{1}_p \mathbf{1}_p'$ and K_{pp} is the commutation matrix (see e.g. Ghazal and Neudecker 2010). As we mentioned in (Rusnačko and Žežula 2015) the following properties of each variance matrix Σ_i hold

$$\mathrm{Tr}\left(\Sigma_i^2\right) = \sigma^4 \left[p + 2\varrho^2 \sum_{j=1}^{p-1} j a_{p-j,i}^2 \right]$$

and

$$\mathbf{1}'\Sigma_i^2\mathbf{1} = \sigma^4 p + 4\sigma^4\varrho R_i + 2\sigma^4\varrho^2 T_i,$$

where

$$T_i = \sum_{j=1}^{p-1} j a_{p-j,i}^2 + 2\left[\sum_{j=2}^{p-2}\sum_{k=1}^{p-j-1} k a_{j,i}a_{p-k,i} + a_1\sum_{j=1}^{p-2}(2j-1)a_{p-j,i} + \right.$$
$$\left. + \sum_{j=2}^{\lceil\frac{p}{2}\rceil-1}\sum_{k=j+1}^{p-j-1}(k-j)a_{j,i}a_{p-k,i}\right] + s_i,$$

and

$$s_i = \begin{cases} 2\sum_{j=1}^{\frac{p}{2}-1} j a_{\frac{p}{2}-j,i}^2 & \text{if } p \text{ is even,} \\ \sum_{j=1}^{\lceil\frac{p}{2}\rceil-1}(2j-1)a_{\lceil\frac{p}{2}\rceil-j,i}^2 & \text{if } p \text{ is odd.} \end{cases}$$

Now we can derive properties of estimators (2.3) and (2.4).

The estimator of parameter σ^2 is clearly unbiased, so its mean square error is equal to its variance and it holds

$$\mathrm{MSE}\left(\widehat{\sigma}^2\right) = \mathrm{Var}\left(\widehat{\sigma}^2\right) = \frac{2\sigma^4}{n^2p^2}\left(p\sum_{i=1}^{t}\frac{n_i^2}{n_i-1} + 2\varrho\sum_{i=1}^{t}\frac{n_i^2}{n_i-1}\sum_{j=1}^{p-1} j a_{p-j,i}^2\right).$$

But the estimator of the parameter ϱ is biased, as usually. To derive approximate properties of this estimator we will use a Taylor expansion. Using this we can find expansions of the mean and the variance of the ratio of two random variables in the form

$$\mathrm{E}\left(\frac{G}{H}\right) = \frac{\mathrm{E}(G)}{E(H)} - \frac{\mathrm{Cov}(G,H)}{\mathrm{E}^2(H)} + \frac{\mathrm{E}(G)}{\mathrm{E}^3(H)}\mathrm{Var}(H) + R_3, \tag{2.5}$$

$$\mathrm{Var}\left(\frac{G}{H}\right) = \frac{\mathrm{Var}(G)}{\mathrm{E}^2(H)} - \frac{2\mathrm{E}(G)}{\mathrm{E}^3(H)}\mathrm{Cov}(G,H) + \frac{\mathrm{E}^2(G)}{\mathrm{E}^4(H)}\mathrm{Var}(H) + R_3, \tag{2.6}$$

where R_3 is the remainder of Taylor series of 3rd order. Using these relations we get that the mean value of estimator (2.4) is

$$\mathrm{E}(\widehat{\varrho}) = \varrho - \frac{2\varrho}{np}\left(\frac{2}{\sum_{i=1}^{t} n_iR_i}\sum_{i=1}^{t}\frac{n_i^2}{n_i-1}R_i - \frac{1}{n}\sum_{i=1}^{t}\frac{n_i^2}{n_i-1}\right)\varrho +$$
$$+ \frac{1}{\sum_{i=1}^{t} n_iR_i}\sum_{i=1}^{t}\frac{n_i^2}{n_i-1}\left(T_i - \sum_{j=1}^{p-1} j a_{p-j,i}^2\right)\varrho^2 -$$
$$- \frac{2}{np}\sum_{i=1}^{t}\frac{n_i^2}{n_i-1}\sum_{j=1}^{p-1} j a_{p-j,i}^2\varrho^3 + O\left(n^{-2}\right),$$

and its variance is

$$\begin{aligned}\operatorname{Var}(\widehat{\varrho}) =& \frac{1}{2\left(\sum_{i=1}^{t} n_i R_i\right)^2}\left[p(p-1)\sum_{i=1}^{t}\frac{n_i^2}{n_i-1} + 4(p-2)\sum_{i=1}^{t}\frac{n_i^2}{n_i-1}R_i\varrho + \right.\\
&+ 2\sum_{i=1}^{t}\frac{n_i^2}{n_i-1}\left(2R_i^2 - 2T_i + \sum_{j=1}^{p-1} j a_{p-j,i}^2\right)\varrho^2 - \\
&- \frac{4\sum_{i=1}^{t} n_i R_i}{np}\left(4\sum_{i=1}^{t}\frac{n_i^2}{n_i-1}R_i - \frac{\sum_{i=1}^{t} n_i R_i}{n}\sum_{i=1}^{t}\frac{n_i^2}{n_i-1}\right)\varrho^2 - \\
&- \frac{8\sum_{i=1}^{t} n_i R_i}{np}\sum_{i=1}^{t}\frac{n_i^2}{n_i-1}\left(T_i - \sum_{j=1}^{p-1} j a_{p-j,i}^2\right)\varrho^3 + \\
&\left. + \frac{8\left(\sum_{i=1}^{t} n_i R_i\right)^2}{n^2p^2}\sum_{i=1}^{t}\frac{n_i^2}{n_i-1}\sum_{j=1}^{p-1} j a_{p-j,i}^2\varrho^4\right] + O\left(n^{-2}\right).\end{aligned}$$

The mean square error of the estimator of parameter ρ is

$$\begin{aligned}\operatorname{MSE}(\widehat{\varrho}) =& \frac{p(p-1)}{2\left(\sum_{i=1}^{t} n_i R_i\right)^2}\sum_{i=1}^{t}\frac{n_i^2}{n_i-1} + \frac{2(p-2)}{\left(\sum_{i=1}^{t} n_i R_i\right)^2}\sum_{i=1}^{t}\frac{n_i^2}{n_i-1}R_i\varrho + \\
&+ \left[\frac{1}{\sum_{i=1}^{t} n_i R_i}\left[\frac{1}{\sum_{i=1}^{t} n_i R_i}\left(\sum_{i=1}^{t}\frac{n_i^2}{n_i-1}\left(2R_i^2 - 2T_i + \sum_{j=1}^{p-1} j a_{p-j,i}^2\right)\right) - \right.\right.\\
&\left.\left. - \frac{8}{np}\sum_{i=1}^{t}\frac{n_i^2}{n_i-1}R_i\right] + \frac{2}{n^2p}\sum_{i=1}^{t}\frac{n_i^2}{n_i-1}\right]\varrho^2 - \\
&- \frac{4}{np\sum_{i=1}^{t} n_i R_i}\sum_{i=1}^{t}\frac{n_i^2}{n_i-1}\left(T_i - \sum_{j=1}^{p-1} j a_{p-j,i}^2\right)\varrho^3 + \\
&+ \frac{4}{n^2p^2}\sum_{i=1}^{t}\frac{n_i^2}{n_i-1}\sum_{j=1}^{p-1} j a_{p-j,i}^2\varrho^4 + O\left(n^{-2}\right)\end{aligned}$$

and the mean square error between the estimators has the form

$$\begin{aligned}\operatorname{MSE}\left(\widehat{\sigma}^2, \widehat{\varrho}\right) =& \frac{2\sigma^2}{np}\left[\left(\frac{2}{\sum_{i=1}^{t} n_i R_i}\sum_{i=1}^{t}\frac{n_i^2}{n_i-1}R_i - \frac{1}{n}\sum_{i=1}^{t}\frac{n_i^2}{n_i-1}\right)\varrho + \right.\\
&+ \frac{1}{\sum_{i=1}^{t} n_i R_i}\sum_{i=1}^{t}\frac{n_i^2}{n_i-1}\left(T_i - \sum_{j=1}^{p-1} j a_{p-j,i}^2\right)\varrho^2 - \\
&\left. - \frac{2}{np}\sum_{i=1}^{t}\frac{n_i^2}{n_i-1}\sum_{j=1}^{p-1} j a_{p-j,i}^2\varrho^3\right] + O\left(n^{-2}\right).\end{aligned}$$

2.2 Maximum likelihood estimators

In this subsection we derive maximum likelihood estimators of unknown parameters. Let us consider one dimensional model from transposed growth curve model in the form $y = W\beta + e$, where $y = \text{vec } Y'$, $W = X \otimes Z'$, $\beta = \text{vec } B'$, $e = \text{vec } \varepsilon'$, $\text{E } e = 0$ and variance matrix Λ of e is of the form (2.1). Its inverse and determinant are in the form

$$\Lambda^{-1} = \begin{pmatrix} I_{n_1} \otimes \Sigma_1^{-1} & 0 & \dots & 0 \\ 0 & I_{n_2} \otimes \Sigma_2^{-1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & I_{n_t} \otimes \Sigma_t^{-1} \end{pmatrix}, \quad \text{and} \quad |\Lambda| = \prod_{i=1}^{t} |\Sigma_i|^{n_i}.$$

Let us denote $\Sigma_i = \sigma^2 F_i(\varrho)$, for $i = 1, 2, \dots, t$ and

$$Q = \begin{pmatrix} I_{n_1} \otimes F_1^{-1}(\varrho) & 0 & \dots & 0 \\ 0 & I_{n_2} \otimes F_2^{-1}(\varrho) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & I_{n_t} \otimes F_t^{-1}(\varrho) \end{pmatrix}.$$

Then, log-likelihood function is of the form

$$\begin{aligned} l(\beta, \sigma^2, \varrho, y) = &- \frac{np}{2} \ln(2\pi) - \frac{np}{2} \ln(\sigma^2) - \frac{1}{2} \sum_{i=1}^{t} n_i \ln |F_i(\varrho)| - \\ &- \frac{1}{2\sigma^2} (y - W\beta)' Q(\varrho)(y - W\beta). \end{aligned} \tag{2.7}$$

Taking derivatives with respect to parameters β, σ^2 and ρ we get equations

$$W'Q(\varrho)W\beta = W'Q(\varrho)y, \tag{2.8}$$

$$np\sigma^2 = (y - W\beta)'Q(\varrho)(y - W\beta), \tag{2.9}$$

$$-\sigma^2 \sum_{i=1}^{t} n_i \text{Tr} \left[F_i^{-1}(\varrho)(A_i - I) \right] = (y - W\beta)' \frac{\partial \, Q(\varrho)}{\partial \varrho} (y - W\beta). \tag{2.10}$$

From the first equation it is clear that the maximum likelihood estimator of parameter β is of the form

$$\widehat{\beta}_{\text{M}} = [W'Q(\varrho)W]^{-1} W'Q(\varrho)y. \tag{2.11}$$

Inserting (2.11) into (2.9) and (2.10) produces a nonlinear system of equations for MLEs of ϱ and σ^2, which we have to solve numerically. Obtained value of ϱ is then inserted into (2.11) to get MLE of β.

Asymptotic variances of these estimators we get from inverse of Fisher information matrix. To get it we need second derivatives of the log-likelihood function (2.7) with respect to unknown parameters σ^2 and ρ. It is well known that for a random vector τ with $\mathrm{E}(\tau) = \mu$ and $\mathrm{Var}(\tau) = \Xi$

$$\mathrm{E}\left(\tau'\Lambda\tau\right) = \mathrm{Tr}(\Lambda\Xi) + \mu'\Lambda\mu. \tag{2.12}$$

Using the previous formula we get

$$\mathrm{E}\,\frac{\partial^2 l(\beta,\sigma^2,\varrho,y)}{\partial\left(\sigma^2\right)^2} = -\frac{np}{2\sigma^4},$$

$$\mathrm{E}\,\frac{\partial^2 l(\beta,\sigma^2,\varrho,y)}{\partial\varrho^2} = -\frac{1}{2}\sum_{i=1}^{t} n_i \mathrm{Tr}\left[F_i^{-1}(\varrho)\left(A_i - I\right)\right]^2,$$

$$\mathrm{E}\,\frac{\partial^2 l(\beta,\sigma^2,\varrho,y)}{\partial\sigma^2\partial\varrho} = -\frac{1}{2\sigma^2}\sum_{i=1}^{t} n_i \mathrm{Tr}\left[F_i^{-1}(\varrho)(A_i - I)\right].$$

Constructing the Fisher information matrix $I_{(\sigma^2,\varrho)}$ we get that asymptotic variances of maximum likelihood estimators of unknown parameters are

$$\mathrm{Var}\left(\widehat{\sigma}^2_{\mathrm{M}}\right) = \frac{1}{2|I_{(\sigma^2,\varrho)}|}\sum_{i=1}^{t} n_i \mathrm{Tr}\left[F_i^{-1}(\varrho)(A_i - I)\right]^2$$

and

$$\mathrm{Var}\left(\widehat{\rho}_{\mathrm{M}}\right) = \frac{np}{2\sigma^4|I_{(\sigma^2,\varrho)}|}.$$

2.3 Comparisons and simulations for the case of two groups

Let us consider the growth curve model with two groups and different decrease of dependency in them. In this section we compare maximum likelihood estimators with estimators based on unbiased estimating equations on the basis of their variances for this case. Some simulations of ratios $\frac{\mathrm{Var}\ \widehat{\sigma}^2}{\mathrm{Var}\ \widehat{\sigma}^2_{\mathrm{M}}} - 1$ and $\frac{\mathrm{Var}\ \widehat{\varrho}}{\mathrm{Var}\ \widehat{\varrho}_{\mathrm{M}}} - 1$ on the admissible interval of positive semi-definiteness of variance matrix for small values of time points (3-6) and different matrices A_1 and A_2, namely we used symmetric Teoplitz matrices with elements $(1,1,\frac{1}{2},\frac{1}{3})$, $(1,1,\frac{1}{2},\frac{1}{4})$, $(1,1,1,\frac{1}{2})$ and $(1,1,1,\frac{1}{3})$, are depicted in the following pictures.

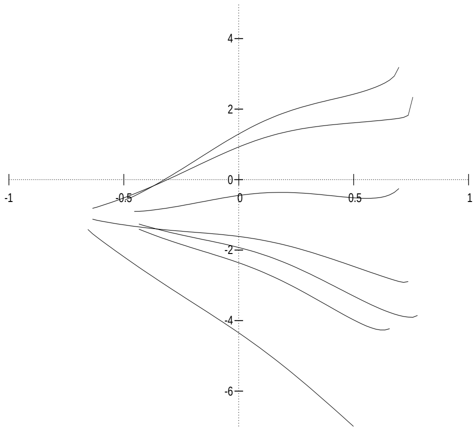

Figure 1: $\frac{\mathrm{Var}\ \widehat{\sigma}^2}{\mathrm{Var}\ \widehat{\sigma}^2_{\mathrm{M}}} - 1$

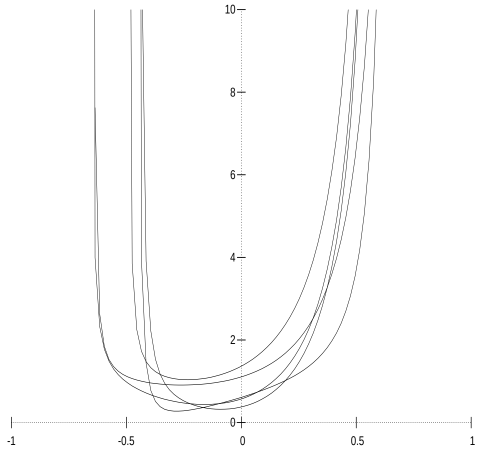

Figure 2: $\frac{\mathrm{Var}\ \widehat{\varrho}}{\mathrm{Var}\ \widehat{\varrho}_{\mathrm{M}}} - 1$

We can see that the maximum likelihood estimator of parameter ϱ is better than the estimator of this parameter based on unbiased estimating equation because all values of function $\frac{\mathrm{Var}\ \widehat{\varrho}}{\mathrm{Var}\ \widehat{\varrho}_{\mathrm{M}}} - 1$ are positive. On the other hand values of function $\frac{\mathrm{Var}\ \widehat{\sigma}^2}{\mathrm{Var}\ \widehat{\sigma}^2_{\mathrm{M}}} - 1$ are in some cases positive and in some cases negative. However, the comparison is not too fair, since we compare the first order approximation of small sample variance on one side with exact asymptotic variance on the other side.

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