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THREE-FACTORIAL EXPERIMENTS GENERATED BY SOME BIB DESIGNS SET UP IN SPLIT-PLOT DESIGNS

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Summary

This paper deals with the construction of some three-factorial experiments carried out in incomplete split-plot designs in which the levels of factors occur as treatments in BIB designs. We consider two cases. In the first case, the levels of factor A are distributed on the whole plots and the combination levels of factors B and C are distributed on the subplots. In the second case, the combination levels of factors A and B are distributed on the whole plots and the levels of factor C are distributed on the subplots. In this paper, the efficiency factors for the main effects of factors and their interaction effects in the inter-block stratum, inter-whole plots stratum and inter-subplots stratum in both cases are given.

Key words and phrases: balanced incomplete block designs, efficiency factors, Kronecker product of matrices, main and interaction effects of factors, split-plot designs, three-factorial experiments

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1. Introduction

Let us consider a three-factorial experiment of split-plot type in which factor A occurs on v_1 levels: A_1, A_2, ..., A_{v_1}, factor B on v_2 levels: B_1, B_2, ..., B_{v_2}

and factor C occurs on v_3 levels: C_1, C_2, ..., C_{v_3}. In this paper, levels are distributed in the same way as treatments in balanced incomplete block (BIB) designs. We consider two cases:

(*i*) the levels of factor A are distributed on the whole plots and the combination levels of factors B and C are distributed on the subplots,

(*ii*) the combination levels of factors A and B are distributed on the whole plots and the levels of factor C are distributed on the subplots.

Case (*i*) was considered by Brzeskwiniewicz and Krzyszkowska (2006).

Let a population of experimental units in an environment be divided into *b* blocks and let each block be additionally divided into: k_1 whole plots while each whole plot is divided into k_2k_3 subplots (case (*i*)), or k_1k_2 whole plots while each whole plot is divided into k_3 subplots (case (*ii*)).

In the next section, we are going to give some basic information about BIB designs and about block designs with incidence matrices equal to the Kronecker product of three incidence matrices of BIB designs. In the third section, the method of the construction of these split-plot experiments and the formulae for the efficiency factors for the main effects and interaction effects are given. Catalogue of these designs for $2 \le r \le 4$ and $2 \le k \le 8$ with $v \le 30$ is included in the fourth section, where *r* is the number of treatment combination replications, *k* is the block size and *v* is the number of treatment combinations. The fifth section shows an example, and in the sixth section, conclusion with discussion are presented.

2. Preliminaries

We are going to use the notion of BIB designs as in

Definition 2.1. (see e.g. Raghavarao, 1971). A balanced incomplete block design (BIB) is an arrangement of v_* treatments in b_* blocks of sizes k_* such that every treatment occurs r_* times and every pair of distinct treatments is contained in exactly λ_* blocks. The numbers v_*, b_*, r_*, k_* and λ_* are called the parameters of BIB design.

Let $d_* = \frac{r_* - \lambda_*}{r_* k_*}$ and let $\mathbf{N}_*$ be the incidence matrix of the above BIB design. Then, $(\mathbf{N}_*, \mathbf{N}_*)$ is also incidence matrix of BIB design with parameters $v_{**} = v_*$, $b_{**} = 2b_*$, $r_{**} = 2r_*$, $k_{**} = k_*$ and $\lambda_{**} = 2\lambda_*$ and

$$d_{**} = \frac{r_{**} - \lambda_{**}}{r_{**}k_{**}} = \frac{r_* - \lambda_*}{r_* k_*} = d_* . \qquad (2.1)$$

Note that: for classical BIB design (i.e. design with $r_* > \lambda_* > 0$) we have $0 < d_* < 1$, for design in which $v_* = b_*$, $r_* = k_* = 1$, $\lambda_* = 0$ (i.e. $\mathbf{N}_* = \mathbf{I}_{v_*}$) we have $d_* = 1$ and for design in which $v_* = k_*$, $r_* = b_* = \lambda_*$, (i.e. $\mathbf{N}_* = \mathbf{J}_{v_* b_*}$) or $v_* = k_*$, $r_* = b_* = \lambda_* = 1$, (i.e. $\mathbf{N}_* = \mathbf{1}_{v_*}$) we have $d_* = 0$.

Ceranka and Goszczurna (1994) give a complete list of incidence matrices of BIB design for $v_* < 20$, $r_* \le 15$, $2 \le k_* \le v_*/2$, $\lambda_* > 0$ and an additional remark about the construction of the designs with $v_*/2 < k_* < v_* - 1$.

Theorem 2.1. Let $\mathbf{N}_A$, $\mathbf{N}_B$, $\mathbf{N}_C$ be incidence matrices of BIB designs with parameters $v_1, b_1, r_1, k_1, \lambda_1$; $v_2, b_2, r_2, k_2, \lambda_2$; and $v_3, b_3, r_3, k_3, \lambda_3$, respectively. Then, block design with incidence matrix

$$\mathbf{N}_1 = \mathbf{N}_A \otimes \mathbf{N}_B \otimes \mathbf{N}_C \qquad (2.2)$$

has the following parameters: $v = v_1 v_2 v_3$, $b = b_1 b_2 b_3$, $r = r_1 r_2 r_3$, $k = k_1 k_2 k_3$ and their association matrix $\mathbf{N}_1\mathbf{N}_1' = \mathbf{N}_A\mathbf{N}_A' \otimes \mathbf{N}_B\mathbf{N}_B' \otimes \mathbf{N}_C\mathbf{N}_C'$ has eigenvalues equal to: $\rho_0 = rk$; $\rho_1 = r_1k_1r_2k_2(r_3 - \lambda_3)$; $\rho_2 = r_1k_1(r_2 - \lambda_2)r_3k_3$; $\rho_4 = (r_1 - \lambda_1)r_2k_2r_3k_3$; $\rho_5 = (r_1 - \lambda_1)r_2k_2(r_3 - \lambda_3)$; $\rho_3 = r_1k_1(r_2 - \lambda_2)(r_3 - \lambda_3)$ $\rho_6 = (r_1 - \lambda_1)(r_2 - \lambda_2)r_3k_3$; $\rho_7 = (r_1 - \lambda_1)(r_2 - \lambda_2)(r_3 - \lambda_3)$ with multiplicities: 1, $v_3 - 1$, $v_2 - 1$, $(v_2 - 1)(v_3 - 1)$, $v_1 - 1$, $(v_1 - 1)(v_3 - 1)$, $(v_1 - 1)(v_2 - 1)$, $(v_1 - 1)(v_2 - 1)(v_3 - 1)$, respectively.

3. Results

In the planning experiments carried out in some split-plot designs, two incidence matrices: $\mathbf{N}_1$ and $\mathbf{N}_2$ are of great importance. Let $\mathbf{N}_1$ be the $(v \times b)$ incidence matrix with respect to blocks, then its (i, j)-element indicates how many times the i-th (i = 1,2,..., v) combination of levels of three factors occurs in *j*-th block (j = 1, ..., b). In this paper, $\mathbf{N}_1$ has the form of (2.2), and therefore, $v = v_1 v_2 v_3$ and $b = b_1 b_2 b_3$. Let $\mathbf{N}_2$ be the ($v \times fb$) incidence matrix with respect to whole plots (inside each block), then, its (i, l)-element indicates how many times the *i*-th combination of levels of three factors (i = 1,2, ..., v) occurs in *l*-th whole plot (l = 1, 2,..., fb), when $f = k_1$ in case (*i*) or $f = k_1 k_2$ in case (*ii*).

For the purpose of observation, we assume after Mejza, Mejza (1984) and Mejza (1986) the randomization linear mixed model. The overall analysis of variance for this experiment is split into so called stratum analyses. These strata are connected with the variability among blocks inside the total experiments, among the whole plots inside blocks and among subplots inside whole plots.

Case (*i*)

Information matrices in these strata are, respectively:

$$\mathbf{C}_1 = \frac{1}{k}\mathbf{N}_1\mathbf{N}_1' - \frac{r}{v}\mathbf{J}_v,$$

$$\mathbf{C}_2 = \frac{1}{k_2}\mathbf{N}_2\mathbf{N}_2' - \frac{1}{k}\mathbf{N}_1\mathbf{N}_1', \tag{3.1}$$

$$\mathbf{C}_3 = r\mathbf{I}_v - \frac{1}{k_2}\mathbf{N}_2\mathbf{N}_2'.$$

The efficiency factors for the contrasts connected with main effects of A, B, C and interaction effects A × B, A × C, B × C and A × B × C in *t*-th stratum (t=1,2,3) are equal to $\varepsilon_{it} = \frac{\mu_{it}}{r}$, respectively, where μ_{it} is *i*-th non-zero eigenvalue of $\mathbf{C}_t$. The order of i = 1,2,…,7 corresponds to: main effect of the factor C, main effect of the factor B, interaction effect of B×C, main effect of the factor A, interaction effect of A×C, interaction effect of A×B, interaction effect of A×B×C, respectively. Formulae for μ_{it} are presented (similarly as in Brzeskwiniewicz, Krzyszkowska, 2001), below. Formula for $\mathbf{C}_1$ in (3.1) implies that $\mu_{i1} = \frac{\rho_{i1}}{k}$, ($i$=1,...,7) are eigenvalues of $\mathbf{C}_1$, when ρ_{i1} are the same as ρ_i in Theorem 2.1. For finding eigenvalues of $\mathbf{C}_2$ and $\mathbf{C}_3$, we consider eigenvalues of $\mathbf{N}_2\mathbf{N}_2'$. It can be seen that $\mathbf{N}_2 = \mathbf{I}_{v_1} \otimes \mathbf{1}_{r_1}' \otimes \mathbf{N}_B \otimes \mathbf{N}_C$ and $\mathbf{N}_2\mathbf{N}_2' = r_1\mathbf{I}_{v_1} \otimes \mathbf{N}_B\mathbf{N}_B' \otimes \mathbf{N}_C\mathbf{N}_C'$.

Then, eigenvalues of $\mathbf{N}_2\mathbf{N}_2'$ are equal to: $\rho_{12} = \rho_{52} = r_1 r_2 k_2 (r_3 - \lambda_3)$; $\rho_{22} = \rho_{62} = r_1(r_2 - \lambda_2) r_3 k_3$; $\rho_{32} = \rho_{72} = r_1(r_2 - \lambda_2)(r_3 - \lambda_3)$; $\rho_{02} = \rho_{42} = r_1 r_2 k_2 r_3 k_3$.

Therefore, $\mu_{i2} = \frac{1}{k_2}\rho_{i2} - \frac{1}{k}\rho_{i1} = \frac{1}{k_2}\rho_{i2} - \mu_{i1}$ and $\mu_{i3} = r - \frac{1}{k_2}\rho_{i2}$.

From the formulae for μ_{it}, we obtain efficiency factors ε_{it} for respective stratum effects which satisfy the inequality $0 \le \varepsilon_{it} \le 1$. They are important, since generally they show which contrasts are estimable (if $\mu_{it} \ne 0$) and with how high efficiency factors in two or three strata. Then, we can select for the analysis of variance this stratum, where the corresponding ε_{it} is the highest.

Note that the efficiency factors in case (*i*) are equal:
in the first stratum:
$\varepsilon_{11}^{(1)} = d_3$, $\varepsilon_{21}^{(1)} = d_2$, $\varepsilon_{31}^{(1)} = d_2 d_3$, $\varepsilon_{41}^{(1)} = d_1$, $\varepsilon_{51}^{(1)} = d_1 d_3$, $\varepsilon_{61}^{(1)} = d_1 d_2$, $\varepsilon_{71}^{(1)} = d_1 d_2 d_3$,
in the second stratum:
$\varepsilon_{12}^{(1)} = \varepsilon_{22}^{(1)} = \varepsilon_{32}^{(1)} = 0$, $\varepsilon_{42}^{(1)} = 1 - d_1$, $\varepsilon_{52}^{(1)} = d_3(1 - d_1)$, $\varepsilon_{62}^{(1)} = d_2(1 - d_1)$, $\varepsilon_{72}^{(1)} = d_2 d_3 (1 - d_1)$,
in the third stratum:
$\varepsilon_{13}^{(1)} = \varepsilon_{53}^{(1)} = 1 - d_3$, $\varepsilon_{23}^{(1)} = \varepsilon_{63}^{(1)} = 1 - d_2$, $\varepsilon_{33}^{(1)} = \varepsilon_{73}^{(1)} = 1 - d_2 d_3$, $\varepsilon_{43}^{(1)} = 0$,
where $d_i = \frac{r_i - \lambda_i}{r_i k_i}$, $i = 1,2,3$.

Case (*ii*)
In this case, information matrices in the strata are, respectively

$$\mathbf{C}_1 = \frac{1}{k}\mathbf{N}_1\mathbf{N}_1' - \frac{r}{v}\mathbf{J}_v,$$

$$\mathbf{C}_2 = \frac{1}{k_1 k_2}\mathbf{N}_2\mathbf{N}_2' - \frac{1}{k}\mathbf{N}_1\mathbf{N}_1',$$

$$\mathbf{C}_3 = r\mathbf{I}_v - \frac{1}{k_1 k_2}\mathbf{N}_2\mathbf{N}_2'.$$

It can be seen that $\mathbf{N}_2 = \mathbf{I}_{v_1 v_2} \otimes \mathbf{1}'_{r_1 r_2} \otimes \mathbf{N}_C$ and $\mathbf{N}_2\mathbf{N}_2' = r_1 r_2 \mathbf{I}_{v_1 v_2} \otimes \mathbf{N}_C\mathbf{N}_C'$. Then, eigenvalues of $\mathbf{N}_2\mathbf{N}_2'$ are equal to: $\rho_{12}^{(2)} = r_1 r_2 (r_3 - \lambda_3)$, $\rho_{22}^{(2)} = r_1 r_2 r_3 k_3$, $\rho_{32}^{(2)} = r_1 r_2 (r_3 - \lambda_3)$, $\rho_{42}^{(2)} = r_1 r_2 r_3 k_3$, $\rho_{52}^{(2)} = r_1 r_2 (r_3 - \lambda_3)$, $\rho_{62}^{(2)} = r_1 r_2 r_3 k_3$, $\rho_{72}^{(2)} = r_1 r_2 (r_3 - \lambda_3)$ with multiplicities n_i, $i = 1,..., 7$.

From the above formulae, we obtain efficiency factors, similarly as in case (*i*):
in the first stratum:
$\varepsilon_{i1}^{(2)} = \varepsilon_{i1}^{(1)}$, $i = 1,..,7$,
in the second stratum:
$\varepsilon_{12}^{(2)} = 0$, $\varepsilon_{22}^{(2)} = 1 - d_2$, $\varepsilon_{32}^{(2)} = d_3(1 - d_2)$, $\varepsilon_{42}^{(2)} = 1 - d_1$, $\varepsilon_{52}^{(2)} = d_3(1 - d_1)$, $\varepsilon_{62}^{(2)} = 1 - d_1 d_2$, $\varepsilon_{72}^{(2)} = d_3(1 - d_1 d_2)$,
in the third stratum:
$\varepsilon_{13}^{(2)} = \varepsilon_{33}^{(2)} = \varepsilon_{53}^{(2)} = \varepsilon_{73}^{(2)} = 1 - d_3$, $\varepsilon_{23}^{(2)} = \varepsilon_{43}^{(2)} = \varepsilon_{63}^{(2)} = 0$.

4. Catalogue of split-plot designs with three factors

In table 1, we present parameters of split-plot designs considered in this paper for $v \le 30$, $2 \le r \le 4$, $2 \le k \le 8$ with efficiency factors ε_{ij} (omitted (1) in ca-se (*i*) and omitted (2) in case (*ii*)). Symbols 1, 2, 3, 4, 5 denote classical BIB designs (see Ceranka, Goszczurna, 1994) with following numbers of treatments in blocks:
(1,2), (1,3), (2,3) for symbol 1,
(1,2), (1,3), (2,3), (1,2), (1,3), (2,3) for symbol 2,
(1,2), (1,3), (1,4), (2,3), (2,4), (3,4) for symbol 3,
(1,2), (1,3), (1,4), (1,5), (2,3), (2,4), (2,5), (3,4), (3,5), (4,5) for symbol 4,
(1,2,4), (1,3,7), (1,5,6), (2,3,5), (2,6,7), (3,4,6), (4,5,7) for symbol 5.

Table 1. Parameters and efficiencies of split-plot designs

								Case (*i*)							**Case (*ii*)**						
no.	N_A	N_B	N_C	v	b	r	k	ε_{11} ε_{12} ε_{13}	ε_{21} ε_{22} ε_{23}	ε_{31} ε_{32} ε_{33}	ε_{41} ε_{42} ε_{43}	ε_{51} ε_{52} ε_{53}	ε_{61} ε_{62} ε_{63}	ε_{71} ε_{72} ε_{73}	ε_{11} ε_{12} ε_{13}	ε_{21} ε_{22} ε_{23}	ε_{31} ε_{32} ε_{33}	ε_{41} ε_{42} ε_{43}	ε_{51} ε_{52} ε_{53}	ε_{61} ε_{62} ε_{63}	ε_{71} ε_{72} ε_{73}
1	1	$\mathbf{I}_2$	$\mathbf{1}_2$	12	6	2	4	0 0 1	1 0 0	0 0 1	0,25 0,75 0	0 0 1	0,25 0,75 0	0 0 1	0 0 1	1 0 0	0 0 1	0,25 0,75 0	0 0 1	0,25 0,75 0	0 0 1
2	2	$\mathbf{I}_2$	$\mathbf{1}_2$	12	12	4	4	0 0 1	1 0 0	0 0 1	0,25 0,75 0	0 0 1	0,25 0,75 0	0 0 1	0 0 1	1 0 0	0 0 1	0,25 0,75 0	0 0 1	0,25 0,75 0	0 0 1
3	3	$\mathbf{I}_2$	$\mathbf{1}_2$	16	12	3	4	0 0 1	1 0 0	0 0 1	0,33 0,67 0	0 0 1	0,33 0,67 0	0 0 1	0 0 1	1 0 0	0 0 1	0,33 0,67 0	0 0 1	0,33 0,67 0	0 0 1
4	1	$\mathbf{I}_3$	$\mathbf{1}_2$	18	9	2	4	0 0 1	1 0 0	0 0 1	0,25 0,75 0	0 0 1	0,25 0,75 0	0 0 1	0 0 1	1 0 0	0 0 1	0,25 0,75 0	0 0 1	0,25 0,75 0	0 0 1
5	1	$\mathbf{1}_3$	$\mathbf{I}_2$	18	6	2	6	1 0 0	0 0 1	0 0 1	0,25 0,75 0	0,25 0,75 0	0 0 1	0 0 1	1 0 0	0 1 0	0 1 0	0,25 0,75 0	0,25 0,75 0	0 1 0	0 1 0
6	$\mathbf{I}_3$	$\mathbf{J}_2$	1	18	18	4	4	0,25 0 0,75	0 0 1	0 0 1	1 0 0	0,25 0 0,75	0 0 1	0 0 1	0,25 0 0,75	0 1 0	0 0,25 0,75	1 0 0	0,25 0 0,75	0 1 0	0 0,25 0,75
7	$\mathbf{I}_2$	1	1	18	18	4	4	0,25 0 0,75	0,25 0 0,75	0,06 0 0,94	1 0 0	0,25 0 0,75	0,25 0 0,75	0,06 0 0,94	0,25 0 0,75	0,25 0,75 0	0,06 0,19 0,75	1 0 0	0,25 0 0,75	0,25 0,75 0	0,06 0,19 0,75
8	$\mathbf{I}_2$	$\mathbf{1}_3$	2	18	12	4	6	0,25 0 0,75	0 0 1	0 0 1	1 0 0	0,25 0 0,75	0 0 1	0 0 1	0,25 0 0,75	0 1 0	0 0,25 0,75	1 0 0	0,25 0 0,75	0 1 0	0 0,25 0,75
9	$\mathbf{1}_2$	1	1	18	9	4	8	0,25 0 0,75	0,25 0 0,75	0,06 0 0,94	0 1 0	0 0,25 0,75	0 0,25 0,75	0 0,06 0,94	0,25 0 0,75	0,25 0,75 0	0,06 0,19 0,75	0 1 0	0 0,25 0,75	0 1 0	0 0,25 0,75
10	$\mathbf{I}_2$	$\mathbf{1}_2$	4	20	20	4	4	0,38 0 0,63	0 0 1	0 0 1	1 0 0	0,38 0 0,63	0 0 1	0 0 1	0,38 0 0,63	0 1 0	0 0,38 0,63	1 0 0	0,38 0 0,63	0 1 0	0 0,38 0,63
11	$\mathbf{I}_4$	$\mathbf{1}_2$	1	24	12	2	4	0,25 0 0,75	0 0 1	0 0 1	1 0 0	0,25 0 0,75	0 0 1	0 0 1	0,25 0 0,75	0 1 0	0 0,25 0,75	1 0 0	0,25 0 0,75	0 1 0	0 0,25 0,75
12	$\mathbf{I}_2$	$\mathbf{1}_4$	1	24	6	2	8	0,25 0 0,75	0 0 1	0 0 1	1 0 0	0,25 0 0,75	0 0 1	0 0 1	0,25 0 0,75	0 1 0	0 0,25 0,75	1 0 0	0,25 0 0,75	0 1 0	0 0,25 0,75
13	$\mathbf{I}_3$	$\mathbf{1}_2$	3	24	18	3	4	0,33 0 0,67	0 0 1	0 0 1	1 0 0	0,33 0 0,67	0 0 1	0 0 1	0,33 0 0,67	0 1 0	0 0,33 0,67	1 0 0	0,33 0 0,67	0 1 0	0 0,33 0,67

c.d. tab. 1

14	$\mathbf{I}_2$	$\mathbf{1}_3$	3	24	12	3	6	0,33 0 0,67	0 0 1	0 0 1	1 0 0	0,33 0 0,67	0 0 1	0 0 1	0,33 0 0,67	0 1 0	0 0,33 0,67	1 0 0	0,33 0 0,67	0 1 0	0 0,33 0,67
15	$\mathbf{I}_4$	$\mathbf{J}_2$	1	24	24	4	4	0,25 0 0,75	0 0 1	0 0 1	1 0 0	0,25 0 0,75	0 0 1	0 0 1	0,25 0 0,75	0 1 0	0 0,25 0,75	1 0 0	0,25 0 0,75	0 1 0	0 0,25 0,75
16	$\mathbf{I}_2$	$\mathbf{1}_4$	2	24	12	4	8	0,25 0 0,75	0 0 1	0 0 1	1 0 0	0,25 0 0,75	0 0 1	0 0 1	0,25 0 0,75	0 1 0	0 0,25 0,75	1 0 0	0,25 0 0,75	0 1 0	0 0,25 0,75
17	$\mathbf{I}_3$	$\mathbf{1}_3$	1	27	9	2	6	0,25 0 0,75	0 0 1	0 0 1	1 0 0	0,25 0 0,75	0 0 1	0 0 1	0,25 0 0,75	0 1 0	0 0,25 0,75	1 0 0	0,25 0 0,75	0 1 0	0 0,25 0,75
18	$\mathbf{I}_3$	1	1	27	27	4	4	0,25 0 0,75	0,25 0 0,75	0,06 0 0,94	1 0 0	0,25 0 0,75	0,25 0 0,75	0,06 0 0,94	0,25 0 0,75	0,25 0,75 0	0,06 0,19 0,75	1 0 0	0,25 0 0,75	0,25 0,75 0	0,06 0,19 0,75
19	$\mathbf{I}_3$	$\mathbf{1}_3$	2	27	18	4	6	0,25 0 0,75	0 0 1	0 0 1	1 0 0	0,25 0 0,75	0 0 1	0 0 1	0,25 0 0,75	0 1 0	0 0,25 0,75	1 0 0	0,25 0 0,75	0 1 0	0 0,25 0,75
20	$\mathbf{I}_2$	$\mathbf{1}_2$	5	28	14	3	6	0,22 0 0,78	0 0 1	0 0 1	1 0 0	0,22 0 0,78	0 0 1	0 0 1	0,22 0 0,78	0 1 0	0 0,22 0,78	1 0 0	0,22 0 0,78	0 1 0	0 0,22 0,78
21	$\mathbf{I}_5$	$\mathbf{1}_2$	1	30	15	2	4	0,25 0 0,75	0 0 1	0 0 1	1 0 0	0,25 0 0,75	0 0 1	0 0 1	0,25 0 0,75	0 1 0	0 0,25 0,75	1 0 0	0,25 0 0,75	0 1 0	0 0,25 0,75
22	$\mathbf{I}_5$	$\mathbf{J}_2$	1	30	30	4	4	0,25 0 0,75	0 0 1	0 0 1	1 0 0	0,25 0 0,75	0 0 1	0 0 1	0,25 0 0,75	0 1 0	0 0,25 0,75	1 0 0	0,25 0 0,75	0 1 0	0 0,25 0,75
23	$\mathbf{I}_3$	$\mathbf{1}_2$	4	30	30	4	4	0,38 0 0,63	0 0 1	0 0 1	1 0 0	0,38 0 0,63	0 0 1	0 0 1	0,38 0 0,63	0 1 0	0 0,38 0,63	1 0 0	0,38 0 0,63	0 1 0	0 0,38 0,63
24	$\mathbf{I}_2$	$\mathbf{1}_3$	4	30	20	4	6	0,38 0 0,63	0 0 1	0 0 1	1 0 0	0,38 0 0,63	0 0 1	0 0 1	0,38 0 0,63	0 1 0	0 0,38 0,63	1 0 0	0,38 0 0,63	0 1 0	0 0,38 0,63

Table of split-plot designs for $v \leq 50$, $2 \leq r \leq 10$, $2 \leq k \leq 10$ with efficiency factors $\varepsilon_{ij}^{(1)}$ are presented by Brzeskwiniewicz and Krzyszkowska (2006). From (2.1), it follows that many designs can have the same parameters and efficiency factors. For example, the design no. 22 in table 1 has the same parameters and efficiency factors as split-plot design with $\mathbf{N}_A = \mathbf{I}_5$ ($v_1 = b_1 = 5,\ r_1 = k_1 = 1,\ \lambda_1 = 0$), $\mathbf{N}_B = \mathbf{1}_2$ ($v_2 = k_2 = 2,\ b_2 = r_2 = \lambda_2 = 1$)

and $\mathbf{N}_C = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \end{pmatrix}$ $(v_3 = 3,\ b_3 = 6,\ r_3 = 4,\ k_3 = 2,\ \lambda_3 = 2)$. In table 1, for the same parameters and efficiency factors, only one design is proposed.

5. Example

We consider design no. 9 in table 1 with parameters $v = 18$, $b = 9$, $r = 4$, $k = 8$. To construct the incidence matrix of this design, we use matrix $\mathbf{1}_2$ and the incidence matrix of the design BIB no. 1 from catalogue of Ceranka, Goszczurna (1994) with parameters $v_* = 3$, $b_* = 3$, $r_* = 2$, $k_* = 2$ and $\lambda_* = 1$. Incidence matrices of these designs have the form:

$$\mathbf{N}_A = \mathbf{1}_2 \qquad (v_1 = k_1 = 2, \qquad b_1 = r_1 = \lambda_1 = 1), \qquad \mathbf{N}_B = \mathbf{N}_C = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$(v_2 = b_2 = v_3 = b_3 = 3,\ r_2 = k_2 = r_3 = k_3 = 2,\ \lambda_2 = \lambda_3 = 1)$.

From $\mathbf{N}_1 = \mathbf{N}_A \otimes \mathbf{N}_B \otimes \mathbf{N}_C$ we can obtain the distribution of levels A, B and C in blocks in both cases:

Case (*i*)

The distribution (before randomization) is schematically given by

block I		block II		block III		block IV		block V		block VI		block VII		block VIII		block IX	
A_1	B_1C_1	A_1	B_1C_1	A_1	B_1C_2	A_1	B_1C_1	A_1	B_1C_1	A_1	B_1C_2	A_1	B_2C_1	A_1	B_2C_1	A_1	B_2C_2
	B_1C_2		B_1C_3		B_1C_3		B_1C_2		B_1C_3		B_1C_3		B_2C_2		B_2C_3		B_2C_3
	B_2C_1		B_2C_1		B_2C_2		B_3C_1		B_3C_1		B_3C_2		B_3C_1		B_3C_1		B_3C_2
	B_2C_2		B_2C_3		B_2C_3		B_3C_2		B_3C_3		B_3C_3		B_3C_2		B_3C_3		B_3C_3
A_2	B_1C_1	A_2	B_1C_1	A_2	B_1C_2	A_2	B_1C_1	A_2	B_1C_1	A_2	B_1C_2	A_2	B_2C_1	A_2	B_2C_1	A_2	B_2C_2
	B_1C_2		B_1C_3		B_1C_3		B_1C_2		B_1C_3		B_1C_3		B_2C_2		B_2C_3		B_2C_3
	B_2C_1		B_2C_1		B_2C_2		B_3C_1		B_3C_1		B_3C_2		B_3C_1		B_3C_1		B_3C_2
	B_2C_2		B_2C_3		B_2C_3		B_3C_2		B_3C_3		B_3C_3		B_3C_2		B_3C_3		B_3C_3

where A_1, A_2 are levels of A; B_1, B_2, B_3 are levels of B; and C_1, C_2, C_3 are levels of C.

The efficiency factors for estimation of effects A, B, C, A×B, A×C, B×C and A×B×C in three strata are the following:

t	$\varepsilon_{1t}^{(1)}$ (C)	$\varepsilon_{2t}^{(1)}$ (B)	$\varepsilon_{3t}^{(1)}$ (B×C)	$\varepsilon_{4t}^{(1)}$ (A)	$\varepsilon_{5t}^{(1)}$ (A×C)	$\varepsilon_{6t}^{(1)}$ (A×B)	$\varepsilon_{7t}^{(1)}$ (A×B×C)
1	0,25	0,25	0,06	0	0	0	0
2	0	0	0	1	0,25	0,25	0,06
3	0,75	0,75	0,94	0	0,75	0,75	0,94

Case (*ii*)

The distribution (before randomization) is schematically given by

block I	block II	block III	block IV	block V	block VI	block VII	block VIII	block IX
$A_1B_1\ \frac{C_1}{C_2}$	$A_1B_1\ \frac{C_1}{C_3}$	$A_1B_1\ \frac{C_2}{C_3}$	$A_1B_1\ \frac{C_1}{C_2}$	$A_1B_1\ \frac{C_1}{C_3}$	$A_1B_1\ \frac{C_2}{C_3}$	$A_1B_2\ \frac{C_1}{C_2}$	$A_1B_2\ \frac{C_1}{C_3}$	$A_1B_2\ \frac{C_2}{C_3}$
$A_1B_2\ \frac{C_1}{C_2}$	$A_1B_2\ \frac{C_1}{C_3}$	$A_1B_2\ \frac{C_2}{C_3}$	$A_1B_3\ \frac{C_1}{C_2}$	$A_1B_3\ \frac{C_1}{C_3}$	$A_1B_3\ \frac{C_2}{C_3}$	$A_1B_3\ \frac{C_1}{C_2}$	$A_1B_3\ \frac{C_1}{C_3}$	$A_1B_3\ \frac{C_2}{C_3}$
$A_2B_1\ \frac{C_1}{C_2}$	$A_2B_1\ \frac{C_1}{C_3}$	$A_2B_1\ \frac{C_2}{C_3}$	$A_2B_1\ \frac{C_1}{C_2}$	$A_2B_1\ \frac{C_1}{C_3}$	$A_2B_1\ \frac{C_2}{C_3}$	$A_2B_2\ \frac{C_1}{C_2}$	$A_2B_2\ \frac{C_1}{C_3}$	$A_2B_2\ \frac{C_2}{C_3}$
$A_2B_2\ \frac{C_1}{C_2}$	$A_2B_2\ \frac{C_1}{C_3}$	$A_2B_3\ \frac{C_2}{C_3}$	$A_2B_3\ \frac{C_1}{C_2}$	$A_2B_3\ \frac{C_1}{C_3}$	$A_2B_3\ \frac{C_2}{C_3}$	$A_2B_3\ \frac{C_1}{C_2}$	$A_2B_3\ \frac{C_1}{C_3}$	$A_2B_3\ \frac{C_2}{C_3}$

The efficiency factors for estimation of effects A, B, C, A×B, A×C, B×C and A×B×C in three strata are the following:

t	$\varepsilon_{1t}^{(2)}$ (C)	$\varepsilon_{2t}^{(2)}$ (B)	$\varepsilon_{3t}^{(2)}$ (B×C)	$\varepsilon_{4t}^{(2)}$ (A)	$\varepsilon_{5t}^{(2)}$ (A×C)	$\varepsilon_{6t}^{(2)}$ (A×B)	$\varepsilon_{7t}^{(2)}$ (A×B×C)
1	0,25	0,25	0,06	0	0	0	0
2	0	0,75	0,19	1	0,25	1	0,25
3	0,75	0	0,75	0	0,75	0	0,75

Comparison of results in the above tables indicates that cases (*i*) and (*ii*) are identical with respect to the estimation (function) of factors A and C effects and for their interaction A×C effects, because their efficiency factors are equal in each stratum. Case (*i*) is more advantageous than case (*ii*) for the estimation of factor B effects, for the interaction of B×C effects, for the interaction of A×B effects and A×B×C effects, because in case (*i*), their efficiency factors in the third stratum (the deeper one) are higher than in case (*ii*).

6. Conclusion and discussion

From the formulae for $\varepsilon_{ij}^{(1)}$ and $\varepsilon_{ij}^{(2)}$ (see Section 3), we can obtain:

Corollary 6.1. In the first stratum, efficiency factors in case (*i*) and (*ii*) are the same, i.e. $\varepsilon_{i1}^{(1)} = \varepsilon_{i1}^{(2)}$ for i = 1,...,7.

Corollary 6.2. In the second and the third stratum, efficiency factors for i = 1, 4, 5 in case (*i*) and case (*ii*) efficiency factors are the same, i.e. $\varepsilon_{ij}^{(1)} = \varepsilon_{ij}^{(2)}$ for j = 2, 3.

Corollary 6.3. In the second stratum, efficiency factors for i = 2, 3, 6, 7 in case (*i*) are at least the same as in case (*ii*), i.e. $\varepsilon_{i2}^{(1)} \le \varepsilon_{i2}^{(2)}$. In the third stratum, the efficiency factors in case (*i*) are higher than in case (*ii*), i.e. $\varepsilon_{i3}^{(1)} \ge \varepsilon_{i3}^{(2)}$ for i = 2, 3, 6, 7. If $d_2 = 1$, then the above efficiency factors are the same, i.e. $\varepsilon_{ij}^{(1)} = \varepsilon_{ij}^{(2)}$ for i = 2, 3, 6, 7 and j = 2, 3. If $d_3 = 0$ then $\varepsilon_{3j}^{(1)} = \varepsilon_{3j}^{(2)}$ for j = 2, 3.

In the planning experiment in split-plot design, there is a problem with the selection of the stratum in which analysis of variance is to be introduced. The decision is simple when the efficiency factors are equal because then, we select the stratum with the higher number since the stratum variance is the smaller the "deeper" the stratum is. However, in case when the above variances are unknown, the choice of stratum is not easy. It may happen that greater efficiency factors in the stratum with a lower number will not balance the smaller variance in the "deeper" stratum with a higher number. Hence, such designs are valuable in which certain $d_i = 0$ or 1, because in certain stratum, some efficiency factors are equal to one. For example:

if $d_1 = 0$ then $\varepsilon_{42}^{(1)} = \varepsilon_{62}^{(1)} = 1$,

if $d_1 = 0$ and $d_2 = 1$ then $\varepsilon_{42}^{(1)} = \varepsilon_{62}^{(1)} = \varepsilon_{42}^{(2)} = \varepsilon_{62}^{(2)} = 1$,

if $d_3 = 0$ and $d_2 = 1$ then

$$\varepsilon_{13}^{(1)} = \varepsilon_{33}^{(1)} = \varepsilon_{53}^{(1)} = \varepsilon_{73}^{(1)} = \varepsilon_{13}^{(2)} = \varepsilon_{33}^{(2)} = \varepsilon_{53}^{(2)} = \varepsilon_{73}^{(2)} = 1.$$

References

Brzeskwiniewicz H., Krzyszkowska J. (2001). Rectangular block designs and certain split-plot experiments. *Colloquium Biometryczne* 31, 69-75.

Brzeskwiniewicz H., Krzyszkowska J. (2006). Some split-plot designs for three-factorial experiments. *Colloquium Biometryczne* 36, 23-40.

Ceranka B., Goszczurna T. (1994). Scheme of balanced incomplete block designs (in Polish). *Colloquium Biometryczne* 24, 42-53.

Mejza I., Mejza S. (1984). Incomplete split-plot designs. *Statist. Prob. Letters* 2, 327-332.

Mejza S. (1986). Experiments in incomplete split-plot designs. (in Polish). *Ann. Pozn. Agricult. Univ. Sc. Diss.* 150.

Raghavarao D. (1971). *Construction and Combinatorial Problems in Design of Experiments*. J. Wiley and Sons, New York.

DOŚWIADCZENIA TRÓJCZYNNIKOWE ZAKŁADANE W UKŁADACH SPLIT-PLOT GENEROWANYCH PRZEZ UKŁADY BIB

Streszczenie

W pracy rozważamy doświadczenia trójczynnikowe założone w niekompletnych układach o pojedynczo rozszczepionych jednostkach eksperymentalnych (split-plot). W tych doświadczeniach poziomy trzech czynników A, B i C są rozmieszczone tak jak obiekty w układach zrównoważonych o blokach niekompletnych (BIB). W pracy rozważamy dwa przypadki rozmieszczenia czynników na dużych i małych poletkach. W pierwszym przypadku poziomy czynnika A rozmieszczamy na dużych poletkach, a kombinacje poziomów czynników B i C na małych poletkach. W drugim przypadku kombinacje poziomów A i B rozmieszczamy na dużych poletkach a poziomy czynnika C na małych. Podajemy współczynniki efektywności dla efektów głównych i interakcyjnych rozważanych czynników w trzech warstwach: blokowej, dużych poletek i małych poletek.

Słowa kluczowe: zrównoważone układy o blokach niekompletnych, współczynniki efektywności, iloczyn Kroneckera macierzy, główne i interakcyjne efekty czynników, układy split-plot, doświadczenia trójczynnikowe

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