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PROBABILISTIC ASPECTS OF RHEOLOGICAL MODELS (PART 1) *

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In scientific considerations of the structure of mechanisms affecting the processed medium almost always there is the necessity of describing this medium in a way appropriate to the description of the mechanism itself. Tasks of this type were posed for solving for a long time. Particularly widely was this range of problems dealt with in soil investigations. Also a similar problem range is connected with the consideration of the working processes of agricultural machines. A working process of machine consists of a series of elementary operations, each of which is realized in a different dynamic system. To gest to know the whole process it is necessary to analyze all the subsequent situations the processes medium enters.

Since rheology is a science dealing with media in such a way that it tries to answer the question of what the deformations and strains are in a given point of the investigated body at a certain moment at known parameters of externam influences and a known history of influences occured earlier, therefore its task is to provide an answer to the question of the physical nature of the medium is. Analizing more closely the above formulation we conclude that since the medium which is processed creates it history by passing through successive stages in which it changes its properties, would it not be of advantage to consider the successive stages separately.

Considering the behaviour of the mass of cereals and other plants, that form jointly a corn-field, in a working process during harvesting, we deal with two closely related questions — a mechanical and a rheological ones. The movement of cereal mass in a cereal combined harvester is described as a system of transfers of the subsequent points the mass in relation to the mechanisms of the machine, which constitutes the

* Part to provide an answer to the question of the physical nature of the work will comprise numerical calculations and will be published in 1979.

mechanical question of the problem. For the rheological part of the problem this movements is determined by the initial and boundary conditions. From the point of view of mechanics the rheological problem will consist in the description of the movement of the points of the mass in relation to one another. Thus the solving of the rheological problem is not possible without the prior solving of the mechanics of movement.

The above presented thesis of the consideration of partial causes and effects is from the mathematical point of view simpler than a global solution. Besides, it seems little probable that it is possible to build a physically realizable mathematical model of cereal mass, at the same time true for such systems as ex. the harvesting, threshing, and cleaning systems of a harvester. As is known from literature, media that are simpler from the mass in question are not well describable mathematically in the sense of rheology. And if we consider that the change speeds of external forcings were low for these media, we have an almost complete set of arguments for, let us call it, the individual, from the point of view of the forcing system, approach to the description of the medium.

Very popular is the making of a description of a body with the help of model analogies and structural models. Ex. a model analogy will be the known in physics soap-bubble of Prandtl or the ideally inflexible homogeneous steel ball. A model analogy can also be manifested in another way, ex. at the building of a diagram of the medium on the bassis of mathematical equations of known phenomena. A structural model is a combination of such components as elements presenting viscosity, elasticity, plasticity.

It should be point out here that simple relationships between these elements concern only homogeneous bodies. And so is someone says that a grain of wheat can be described by a structural model equivalent ex. to the model of Bingham, the recipient of this information will understand that it is a homogeneous body. But this is not so. However, it may be true to say that the Bingham model can be used to approximating the description of the body in question. The difference between the two statements is considerable and for its filling we should turn to the methods of micro-rheology.

Micro-rheology, dealing with bodies with a structure, has two basic methods: structural analysis, and structural theory. The first concerns experimental analysis, and this serves to form an appropriate hypothesis, which is then justified in the structural theory. This theory assumes a certain summing up of the properties of the component elements.

Thus we should build a theoretical structural model the components of which would be theoretical bodies, i.e. at least quasi-homogeneous. It seems, that because of the inaccessibility of the methods of structural analysis a grain can be, at the most, approximated with a mono structural model, while cereal mass could be described on the basis of the structural theory, that is basing on equations of mathematical physics. This follows from the fact that cereal mass has a large number of components, each of which can be described by a mono structural model. A mono model will in this case be a theoretical body. This then constitutes a bridge between a poly structural model and the structural theory (ex. a hydrodynamic one considering cracks). The accuracy of the description depends in this situation on the technological requirements.

If then it is possible to utilize the apparatus of physical equations, then, because of the probabilistic character of the medium, it is possible to utilize the methods of object identification.

At present the situation in rheology is still similar to that of Mendelejev during building his periodic table of elements. He provided empty spaces for undiscovered elements. The table of rheological bodies is filled at its ends with classical bodies, towards the centre with theoretical bodies, and the very centre is a blank space. This is the space for physical media. Identification must then consist in the finding of an appropriate space for a given medium in the table. The methods of identification allow for the determination of two things: the structure, and the random interferance of the picture of this structure. It is wort noticing that this second component is often interpreted as a random margin, an unknown element.

Below we present a sketch of a probabilistic method based on the structural theory. Here the Wiener's theory of the "fourth box" is applied.

$$\vec{y} = K\vec{x}$$

where:

K — opertator dependent from time,

$\vec{x}$ — vector of coercion,

$\vec{y}$ — vector of the reaction of material.

The operator K must meet the following conditions

1) linearity — $K(a_1x_1 + a_2x_2) = a_1Kx_1 + a_2Kx_2$

where:

a_1, a_2 — arbitrary values,

2) implication — $[y(t) \to 0 \wedge t \to \infty] \Rightarrow [x(t) \to 0]$

3) relation between the operator K and the function of transfer $\Phi(\omega)$

$$K(Ie^{i\omega t}) = \Phi(\omega)\, I e^{i\omega t}$$

where:

I — unit vector,

$\Phi(\omega)$ — matrix of the components $\varphi_{ik}(\omega)$.

$$K_{ij}(\xi) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \varphi_{ij}(\omega) e^{i\omega\xi} d\omega, \tag{2}$$

$$K_{ij}(\xi) = 0, \ \xi < 0,$$

$$K_{ij} f(t) = \int_{0}^{+} k_{ij}(t-\tau) f(\tau) d\tau, \ t > 0 \tag{3}$$

The operator K is a matrix of the components k_{ij} presented by the convolution (3) where k_{ij} are Fourier's transforms (2) of the components of the matrix of function of transfer. The functions k_{ij} are of the class of generalized functions. The transfer function (transmittance of system) has the form

$$\Phi(i\omega) = \frac{P_m(i\omega)}{Q_n(i\omega)} \tag{4}$$

where:

P and Q are polynominals of the m and n degrees in effect of substituting the operators of differentiating of the right (x) and left (y) sides of differential equation respectively with $i\omega$ (Laplace's transformations).

The expression KI occuring in the third condition is the reaction of the system to Dirac's impuls and the transfer function describes the reaction of the medium to sinusoidal coercion.

The determination of the operator K is made with the help of Laplace's transformations of the functions x and y. The functions are obtained in m successive experiments.

Expanding the third condition we obtain the formal relations determining the operator K.

$$Ly_j(z) = Lk_j(z)\, Lx_j(z) \tag{5}$$

where:

L — Laplace's transformation,
z — complex variable,
j — 1, 2, ..., m number of experiments.

$Ly_j(z) = \|Ly_{ij}(z)\|$ (matrix from terms of particular realizations for
$i = 1, 2, ..., n$ i degrees of freedom)

$$ay_{ij}(z) = \int_{0}^{\infty} y_{ij}(\tau) e^{-z\tau} d\tau,$$

$$ax_j(z) = \| ax_{ij}(z) \|,$$

$$ax_{ij}(z) = \int_0^\infty x_{ij}(\tau) e^{-z\tau} d\tau,$$

$$ak_j(z) = \| ak_{ij}(z) \|,$$

$$ax_{ij}(z) = \int_0^\infty k_{ij}(\tau) e^{-z\tau} d\tau,$$

ax_j and ay_j — vectors.

Thus the relation (5) presents a matrix of nxm unknown values $Lk_{ij}(z)$ determined by the values $Lx_{ij}(z)$ and $Ly_{ij}(z)$. Applying now to $Lk_{ij}(z)$ the reverse transformations of Laplace we obtain $K_{ij}(z)$. The presented method can be applied for non-stationary systems.

If the object is stationary the formalism is simplified to

$$Fy_j(\omega) = \Phi(\omega) Fx_j(\omega) \tag{6}$$

where: F — Fourier's transform

$$Fy_j(\omega) = \| Fy_{ij}(\omega) \|,$$

$$Fy_{ij}(\omega) = \int_{-\infty}^{\infty} y_{ij}(\tau) e^{-i\omega\tau} d\tau,$$

$$Fx_j(\omega) = \| Fx_{ij}(\omega) \|,$$

$$Fx_{ij}(\omega) = \int_{-\infty}^{\infty} x_{ij}(\tau) e^{-i\omega\tau} d\tau.$$

From m experiments we find the sought nuclei $\varphi_{ij}(\omega)$ (2), If $m = n = 1$ then it is particularly worthwhile to consider two cases

$$K_{11}(\xi) = a_n \delta^{(n)}(\xi),\ n = 0,1,\ldots,l, \tag{7}$$

$$K_{11}(\xi) = a\delta(\xi) + k_{11}^{0}(\xi), \tag{8}$$

where:

$\delta^{(n)}(\xi)$ — n-th derivative of delta function,

a_n — constants,

$k^0{}_{11}(\xi)$ — function of limited oscillation.

The first case is reduced to the equation

$$y_1 = a_l \frac{d^l x_1}{dt^l} + a_{l-1} \frac{d^{l-1} x_1}{dt^{l-1}} + \ldots + a_0 x_1. \tag{9}$$

The second case gives

$$y_1 = a_0 x_1 + \int_0^t K_{11}^0 (t - \tau) x_1(\tau) d\tau. \tag{10}$$

This case, determining the relation of ex. strain (σ), deformation (ε), is reduced to the finding fo the nucleus $k_{11}(\xi)$ or the corresponding spectral density $\varphi_{11}(\omega)$.

The relation (10) allows for the utilization of the information theory in the identification of a body, and k_{11} plays here the role of the memory of a filter. Of course the error in the evaluation of the model of body will be the lower the narrower the transmission band of the filter, and the closer to sinusoid the signal $x(t)$.

Ex.

$$y(t) = y_0(t) + \varepsilon y_1(t), x(t) = x_0(t) + \varepsilon x_1(t), \varepsilon \ll 1 \tag{11}$$

where:

y_0, x_0 — determined functions,

y_1 and x_1 — random stationary functions.

For the determination of the transfer function Φ it is necessary to determine

$$k(\xi) = k_0(\xi) + \varepsilon k_1(\xi) \tag{12}$$

where:

k_0 — determined part,

k_1 — random part.

Using (6) we get

$$f(t) = y_1(t) - y_0(t) = y_1(t) - \int_{-\infty}^{t} k_0(t - \tau)\, x_0(\tau)\, d\tau \tag{13}$$

$$k_1(\xi) = \int_{-\infty}^{\infty} G(\xi - \tau)\, f(\tau)\, d\tau \tag{14}$$

where:

$$G(\xi - \tau) = \frac{1}{2\pi} \int_{-\infty}^{t} \frac{e^{iw(\xi - \tau)}}{Fx_0(w)} dw.$$

$$Fx_0(w) = \int_{-\infty}^{t} x_0(\tau)\, e^{-iw\tau} d\tau.$$

The expected value $k_1(\xi)$ is

$$Ek_1(\xi) = \int_{-\infty}^{\infty} G(\xi - \tau) Ef(\tau)\, d\tau \tag{15}$$

The correlation function for $k_1(\xi)$ has the form

$$K(t_1, t_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(t_1 - \tau) G(t_2 - \tau_2) E[f(\tau_1) - Ef(\tau_1)] [f(\tau_1) - Ef(\tau_2)] \, d\tau_1 \, d\tau_2. \tag{16}$$

In this way for the case of normality of the process the problem, after the calculation of the first two moments (15 and 16), and after the determination of the transfer function of the system, is fully solved.

The transfer function determined the form of the differential equation connecting ex. strain (y) and deformation (x) in time, while the two calculated moments allow to predict the probability of the realization of this model. One more thing requires attention in this action: the entrance (x) and the exit (y) are presented as sums of the determined and the random parts. Now this determined part is just the element of the structural analysis, a very important thing from the point of view of the possibility of getting to know this process. It is just the transfer function of the system that determines it.

Continuing further the problem range of the structural analysis it seems worthwhile to point to the equation

$$\frac{d^2y(t)}{dt^2} + \omega_0^2 y(t) = \mu f\left[y(t), \frac{dy(t)}{dt}\right] + \mu \nu x(t) \tag{17}$$

where:

$x(t)$ — random stationary differentiable process

$$\nu \geqslant 0.$$

This equation is important because in its part $\mu f\left[y(t), \frac{dy(t)}{dt}\right]$ it contains the problem of movement damping. The problem of friction is widely discussed in rheology.

Equation (17) can now be solved also for the case when $x(t)$ is a non--stationary function (by using the function of spectral density dependent from time).

This case, although very important from the practical point of view, does not bring any particularly important moments to calculation schemes. Below we present a sketch of solution for a case when the density of probability of solutions of the equation is not normal. A considerable attention is paid in the mechanics of agricultural machines coercions, and it is often assumed that they must be normal. But from the experimental point of view it does not necessarily have to be so. The normal process takes place when the effect occurs as result of the summing up of infinitely many uncorrelated increases. A process of such increases is not differentiable. Thus the process $x(t)$ is influenced by the number of components that can be correlated to one another. This is the cause for accepting an anormal distribution for x.

The equation (17) is apart from that nonlinear, because of the element of friction. The solving should be started from linearization. Linearization is done by decomposing all the nonlinear functions included in a given equation into Taylor's string in respect to fluctuation. Let's adopt

$$y = y_1, \quad \dot{y}_1 = y_2 \tag{18}$$

which then gives

$$\dot{y}_2 + \omega_0^2 y_1 = \mu f[y_1, y_2] + \mu \nu x \tag{19}$$

$$\dot{y}_1 = y_2.$$

which can be presented as

$$\frac{dy_i}{dt} = F_i(y_1, y_2, x, \nu, \mu) \tag{20}$$

$i = 1, 2$
$\mu = \text{const.}$
$\nu = \text{const.}$

Linearization should be done in respect of the value

$$x^0 = x - m_x, \quad \overset{0}{y_i} = y_i - m_{yi} \tag{21}$$

m — mean value.

Considering the fact that mean value constitutes a main part of the function for the first two terms of the Taylor's string, we obtain

$$F_i[y_1, y_2, x, \mu, \nu] = F_i[m_{y_1} + \overset{0}{y_1}, m_{y_2} + \overset{0}{y_2}, m_x + x^0, \mu, \nu] \simeq$$

$$\simeq F_i[m_{y_1}, m_{y_2}, m_x, \mu, \nu] + \sum_{j=1}^{2} \frac{\delta F_i}{\delta m_{y_j}} \overset{0}{y_j} + \frac{\delta F_i}{\delta m_x} x^0. \tag{22}$$

Since

$$m_{\overset{0}{y_i}} = 0, \qquad m_{x^0} = 0 \tag{23}$$

and applying to the obtained equations the operation of average values we obtain the final form of the equation

$$\ddot{y} - \mu \frac{\delta}{\delta m_{\dot{y}}} \left\{ f \left[m_y, m_{\dot{y}} \right] \right\} \dot{y} - \left(\mu \frac{\delta}{\delta m_y} \left\{ f \left[m_y, m_{\dot{y}} \right] \right\} - \omega_0^2 \right) y -$$
$$- m_{\ddot{y}} + \mu \frac{\delta}{\delta m_{\dot{y}}} \left\{ f \left[m_y, m_{\dot{y}} \right] \right\} m_{\dot{y}} + \mu \frac{\delta}{\delta m_y} \left\{ f \left[m_y, m_{\dot{y}} \right] \right\} m_y -$$
$$- \mu f [m_y, m_{\dot{y}}] = \mu \nu x. \tag{24}$$

The above equation can be treated in two ways: as an equation of constant coefficients (assuming that the mean value from experiment is a determined function) or that the equation has random coefficients (in the population of experiments the mean value is a random function).
For $t = 0$

$$y = y_0 + y_I \tag{25}$$

where:
y_I — particilar solution of heterogeneous equation

$$y_0 = \sum_{j=1}^{2} c_j y_j,$$

c_j — constants determined from the preliminary conditions of given distribution densities,
y_j — system of linearily independent solutions of homogeneous equation.
Obviously

$$y = c_1 y_1 + c_2 y_2 + y_I.$$

Marking

$$Q_1 = -\mu \frac{\delta}{\delta m_{\dot{y}}} \{ f [m_y, m_{\dot{y}}] \},$$

$$Q_2 = -\mu \frac{\delta}{\delta m_y} \{ f [m_y, m_{\dot{y}}] - \omega_0^2 \},$$

$$H = m_{\ddot{y}} - \mu \frac{\delta}{\delta m_{\dot{y}}} \left\{ f \left[m_y, m_{\dot{y}} \right] \right\} m_{\dot{y}} - \mu \frac{\delta}{\delta m_y} \left\{ f \left[m_y, m_{\dot{y}} \right] \right\} m_y + \mu f \left[m_y, m_{\dot{y}} \right].$$

We obtain

$$\ddot{y} + Q_1 \dot{y} + Q_2 y = \Psi [\mu, \nu, H, x] \tag{26}$$

The function H should be treated as a stochastic signal usable (part that is not the white noise). It may be that H will be the only component of the function ψ. The above remark is significant because it makes the notion of stochastic complex coercion more precise.

Since y_j are linearily independent solutions of homogenous equation the following conditions must occur

$$P\{|a_1 y_1 + a_2 y_2| = 0\} = 1, \tag{27}$$

$$E\{|a_1 y_1 + a_2 y_2|\} = 0. \tag{28}$$

where:

a_1, a_2 — constants.

The problem then has two preliminary conditions y (0) and $\dot{y}$ (0) having their own distributions, and so at the determination of the distribution function of equation (24) there is the necessity of considering the distribution of constants.

There is a considerably large group of media characterized by a high regularity, for which coercion is negligible, and the preliminary conditions are very important. For such media the distributions of solutions of their equations will be a function of the preliminary conditions. It seems to the point to take a closer look at these solutions. The problem of the determination of the probability density of solutions of the movement equations seems as follows:

$$\dot{y}_j = F_j[s, y_1, y_2], \quad j = 1,2 \tag{29}$$

$$y_j(0) = y_j^0(\gamma) \tag{30}$$

where: $y_j^0(\gamma)$— random variables of a known total distribution of probability.

γ is here an elementary occurence. Basing on the mathematical description of elementary occurence we should indicate what is meant by it in a concrete case. Then we should find the transformation of random wector y^0 (γ) — describing the preliminary conditions) into the vector $\vec{y}$ (t, γ) — describing the solution of differential equation.

$$y_j(t,\gamma) = g_j[t = t_0, t, y_1^0(\gamma), y_2^0(\gamma)] \tag{31}$$

where:

g_j — the sought deterministic functions. Now if g_j for every j will be a unequivocal solution of the preliminary problem (29) and (30) in which the preliminary conditions have the total probability density $f_0[\overset{0}{y_1}(\gamma), \overset{0}{y_2}(\gamma)]$ and the functions will be continuous and will have continuous derivatives $\frac{\delta g_j}{\delta \overset{0}{y_k}(\gamma)}$, $k = 1, 2$, then the sought density of probability will be

$$f[t; y_1, y_2] = f_0[y_1^0(\gamma), y_2^0(\gamma)] \left| \frac{\delta(h_1, h_2)}{\delta(y_1, y_2)} \right|, \tag{32}$$

where: $h_j = [g_j]^{-1}$ (reciprocal of function g), i.e., for every t it is

$$y_1^0(\gamma) = h_j[t; y_1(\gamma), y_2(\gamma)]. \tag{33}$$

If however the system of equations (29) has a complex form and it is not possible to obtain explicit solutions, the probability density can be determined otherwise. The Liouville-Gib's equation is applied and we obtain the distribution

$$f[t; y_1, y_2] = f_0[y_1^0(\gamma), y_2^0(\gamma)] \exp\left\{ -\int_{t_0}^{t} \left[\frac{\delta F_1(\tau, t_0; g_1, g_2)}{\delta g_1} + \right.\right.$$
$$\left.\left. + \frac{\delta F_2(\tau, t_0; g_1, g_2)}{\delta g_2} \right] d\tau \right\}. \tag{34}$$

In order to obtain the distribution (34) certain conditions must be met.

Let the point M_0 of the co-ordinates $(\overset{0}{y_1}, \overset{0}{y_2}) = \vec{y}^0$ be the initial point. $\vec{Y}^0$ will be the realization of the random vector $\vec{y}^0(\gamma)$ of the preliminary conditions. The probability that M_0 belongs to a certain set S_0 in phase space

$$P\{M_0 \in S_0\} = \iint_{S_0} f_0(y_1^0, y_2^0)\, dy_1^0, dy_2^0, \tag{35}$$

where:

$f_0(y_1^0, y_2^0)$ is the joint density function of the initial conditions, about which we should assume that it is given. So there arises the question where do we take it from. Now there are two methods mutualy complementary, through investigating experimentally the material constants and through mathematical modelling. For the time $t > t_0$ it will be

$$P\{M_t \in S_t\} = \iint_{S_t} f(y_1, y_2)\, dy_1, dy_2 \tag{36}$$

where: $f(y_1, y_2)$ — joint function of the density of random variables y_1 and y_2 at the moment t.

Since movement is ruled by the deterministic laws

$$\iint\limits_{S_o} f_o(y_1^\circ, y_2^\circ)\, dy_1^\circ, dy_2^\circ = \iint\limits_{S_t} f(y_1, y_2)\, dy_1, d_{y2}. \tag{37}$$

Hence we obtain

$$\frac{\delta f}{\delta t} + \frac{\delta f[F_1(y_1, y_2 x, \mu, v)]}{\delta y_1} + \frac{\delta f[F_2(y_1, y_2, x, \mu, \gamma)]}{\delta y_2} = 0. \tag{38}$$

Solving the equation (38) we obtain the given above result (34). The equation (38) was given, not because from it is easier to obtain the distribution than in the case of (32) but because, it is valid in the statistical theory of irreversible processes. Apart from that, as it was already mentioned in the case of difficulties in solving the equations (29), the equation (38) is easier for numerical solving.

In the case when a body or medium we investigate is not characterized by a considerable regularity, and such a medium is staw mass, and coercion is not normal for the finding of the distribution density of the solution of differential equation (17) we can utilize Edgenworth's string.

$$f(y|t) = \varphi(y|t) - \frac{1}{3!}\frac{\mu_3}{\sigma^3}\varphi^{(3)}(y|t) + \frac{1}{4!}\left(\frac{\mu_4}{\sigma^4} - 3\right)\varphi^{(4)}(y|t) + \frac{10}{6!}\left(\frac{\mu_3}{\sigma^3}\right)^2 \varphi^{(6)}(y|t) -$$

$$-\frac{1}{5!}\left(\frac{\mu_5}{\sigma^5} - 10\frac{\mu_3}{\sigma^3}\right)\varphi^{(5)}(y|t) - \frac{35}{7!}\frac{\mu_3}{\sigma^3}\left(\frac{\mu_4}{\sigma_4} - 3\right)\varphi^{(7)}(y|t) - \frac{280}{9!}\left(\frac{\mu_3}{\sigma^3}\right)\varphi^{(9)}(y|t) + \ldots \tag{39}$$

where:

φ — normal distribution,
μ_ν — central moment,
σ — standard deviation.

$$\mu_\nu[y_I(t)] = \int_0^t \ldots \int_0^t \frac{\begin{vmatrix} y_1(\dot\tau) & y_2(\dot\tau) \\ y_1(\tau) & y_2(\tau) \end{vmatrix}}{\begin{vmatrix} y_1(\dot\tau) & y_2(\dot\tau) \\ \dot y_1(\tau) & \dot y_2(\tau) \end{vmatrix}} \cdot \frac{\begin{vmatrix} y_1(\ddot\tau) & y_2(\ddot\tau) \\ y_1(\tau) & y_2(\tau) \end{vmatrix}}{\begin{vmatrix} y_1(\ddot\tau) & y_2(\ddot\tau) \\ \dot y_1(\tau) & \dot y_2(\ddot\tau) \end{vmatrix}} \cdot \;\ldots$$

$$\cdot \frac{\begin{vmatrix} y_1(\check\tau) & y_2(\check\tau) \\ y_1(\tau) & y_2(\tau) \end{vmatrix}}{\begin{vmatrix} y_1(\check\tau) & y_2(\check\tau) \\ \dot y_1(\check\tau) & \dot y_2(\check\tau) \end{vmatrix}} K_x(\dot\tau, \ddot\tau, \ldots, \check\tau)\, d\dot\tau d\ddot\tau \ldots d\check\tau \tag{40}$$

where: K — correlation function.

The distribution (39) is different from normal, and the difference is greater as higher are the values of higher moments.

Now we should answer the question how to make more precise the non-linear element occuring in equation (17). This element can be modelled in the way presented below.

We must assume that the friction force T depends on the relative speed V of bodies in friction. On every elementary surface dA of the bodies in contact of the surface A_c there occurs the unit tangent force $\bar{\tau}$. It occurs at determined time moments, and we can assume that between these moments its value is zero. For the sake of generality of considerations we may assume that between these momentary points the values of the unit forces may change their denominations into reverse. The part of surface on which $\bar{\tau} > 0$ is the surface of actual contact A_r at a given moment.

So $A_c > A_r$

$$T = \int_{A_c} \bar{\tau}_c dA = \int_{A_r} \bar{\tau}_r dA \tag{41}$$

$\bar{\tau}_r$ — mean value of unit force in the area of possible contact of two bodies,

$\bar{\tau}_c$ — momentary mean unit value (actual).

$$\tau_c = \frac{T}{A_c}, \quad \tau_r = \frac{T}{A_r}, \quad T = A_r \tau_r = A_c \tau_c. \tag{43}$$

If we assume that the relations

$$\frac{A_r}{A_c} = \frac{\tau_c}{\tau_r} = p(V) \tag{44}$$

present a dimentionless statistical characterization of friction, it is possible to interpret them as the probability that all the points comprised by a given contour are media of transmission and transformation of energy. Vector of speed $\vec{V}$ can be discomposed into two ortogonal components

$$\vec{V} = V_x + V_y.$$

Thus

$$p(V) = p(V_x)\, p(V_y/V_x). \tag{45}$$

The conditional probabilities V_x and V_y are independent e.g. $p(V_y/V_x) = p(V_y)p(V_x)$. In the general case $p(V_y/V_x) \neq p(V_y)$. However for straw mass we can accept the independence of directions. In order to assign numerical values to the relation (45) we should find the function Ψ $(V_x;$ $V_y)$ fulfilling the equation

$$p\,(V_y/V_x) = \Psi(V_x;\, V_y)\, p\,(V_y),$$

or

$$p\,(V) = \Psi\,(V_x;\, V_y)\, p\,(V_x),\, p\,(V_y). \tag{46}$$

The function Ψ must be positive and finite. Let $\overline{\Psi}$ be the upper limit of Ψ

$$\tilde{\Psi} = \sup \Phi(V_x; V_y)$$

then

$$p(V) < F(v) = \tilde{\Psi} p(V_x) p(Vy). \tag{47}$$

Since $|v| = \sqrt{v_x^2 + v_y^2}$ then

$$F\left(\sqrt{v_x^2 + v_y^2}\right) = \tilde{\Psi} p(V_x) p(V_y).$$

The solution of this function equation is the function.

$$p(V_x) = \eta e^{aV_x^2}. \tag{48}$$

If $p\,(V_x) \leqslant \alpha$ then the indetermined coefficient a should be negative and have the value

$$a = -\alpha^2$$

and hence

$$p(V_x) = \eta e^{-\alpha^2 V_x^2} = \frac{A_r}{A_c} = \frac{\tau_c}{\tau_r}. \tag{49}$$

From the last equation we find

$$T = A_r \tau_r = A_c \tau_c = \eta A_c \tau_r e^{-\alpha^2 V^2 x}. \tag{50}$$

The friction coefficient, at a known normal force N, and for the value $p_n = \dfrac{N}{A_c}$ (unit loading on the possible surface of contact) will be

$$f_I = \eta \frac{\tau_r}{p_n} e^{-\alpha^2 V^2}. \tag{51}$$

The values τ_r can be determined on the basis of the changes of kinetic energy of material body. Now this energy during movement changes into heat, which is in the contact layer of the thickness h. Considering the statistical character of the process of dissipation of energy of bodies in friction we substitute the surface densities with their mean value

$$\rho = \frac{\rho_1 + \rho_2}{2}. \tag{52}$$

Assuming that the momentary tangent unit force is equal to the change rate of unit kinetic energy, we get

$$\tau_r = \frac{d}{d_s}\left(\frac{\rho_1 V^2}{2}\right) = \frac{1}{2}\frac{d\rho_1}{ds} V^2 + \rho_1 V \frac{dV}{ds}.$$

Assuming further $\rho^0 = \frac{d\rho}{ds}$ and $\tau_0 = \rho_1 \frac{dV}{dt} - \frac{1}{2}\left(\frac{d\rho_2}{ds}\right) V^2$

we finally get

$$\tau_r = \varrho' V^2 + \tau_o \tag{53}$$

and

$$f_r = \frac{\eta}{p_n}(\tau_0 + \rho^0 V^2) e^{-\alpha^2 V^2}. \tag{54}$$

In order to determine the friction coefficient more accurately we should take a closer look both at the surface phenomena accompanied by local changes of temperature, and at the movement of the medium which fills the space between the two bodies in friction. A characterization of this medium is, in the case of cereal mass, almost impossible. So there remain model, ex. analogue, investigations. Assuming that resistance is viscous the value of the additional force will depend, apart from speed, on the thickness of the contact layer h and the surface area of possible contact A_c, and will be

$$T_1 = \xi \frac{1}{h} A_c v \tag{55}$$

$$f_1 = \xi \cdot \frac{1}{h} \cdot \frac{A_c}{N} V = \frac{\xi}{h p_n} V.$$

And hence we finally get

$$f_T = \frac{\eta}{p_n}(\tau_0 + \rho' V^2) e^{-\alpha^2 V^2} + \frac{\xi}{h p_n} V.$$

For a still fuller picture we should expand the formula (56) with elements containing term for the maximum actual pressure on the surface and the temperature of the medium between the bodies and the temperature of the surface itself.

Accurate solutions can be obtained through solving the equations of conductivity in elastic medium at assumed peripheral conditions resulting from the mechanics of movement of the medium.

The composition of the friction element as to both the interial and exterial friction in actual conditions is complex and as it was presented in the preceding two examples it may involve different difficulties of mathematical nature. Often then it is profitable to simplify, even considerably, the model, so as to obtain a solution with the help of which we can expand the notion apparatus for further expantion of the problem. As an example we can use here the model of friction propsed by Van der Pol. Despite the fact that many years have passed since the moment of its creation it is still, as can be seen from literature, utilized in problems often very remote from friction. The equation system (19) equivalent to the homogeneous equation (17) will take, for the Van der Pol case, the form

$$\dot{y}_1 = y_2$$
$$\dot{y}_2 = -y_1 - \varepsilon\,(y_1^2 - 1)y_2 \tag{57}$$

where:

$$\omega_0^2 \equiv 1$$
$$\mu \equiv -\varepsilon$$
$$f[y_1, y_2] \equiv (y_1^2 - 1)y_2.$$

We make the following approximation

$$y_1^2 y_2 \simeq a_\circ + a_1 y_1 + a_2 y_2 \tag{58}$$

where: a_1, a_2 constants chosen in such a way that

$$E(y_1^2 y_2 - a_\circ - a_1 y_1 - a_2 y_2)^2 = \text{min.} \tag{59}$$

Solving the presented variation problem we get

$$a_\circ = E[(y^2{}_1 - 1)y_2] - a_1 E(y_1) - a_2 E(y_2) \tag{60}$$

$$a_{1,2} = \frac{E\{(y_1^2 - 1)\,y_2 - E[(y_1^2 - 1)y_2] \oplus [y_1 - E(y_1)]}{[y_1 - E(y_1)] \oplus [y_1 - E(y_1)]} \tag{61}$$

$a_{1,2}$ — vector of constants,
$\oplus$ — Kronecker's product of vectors.

Hence

$$a_0 = 0, \; a_1 = 0, \quad a_2 = \frac{E(y_1^2 \, y_2^2)}{E(y_2^2)}. \tag{62}$$

Now, passing on to the heterogeneous equation let us assume for the sake of simplicity (otherwise than before) that coercion is a normal stationary process of mean value equal zero.

If $\varepsilon \ll 1$ then y_1, y_2 are normal processes, and so

$$E(y_1^2 y_2^2) = E(y_1^2) E(y_2^2) \tag{63}$$

(y_1, y_2 must be independent).

From (63) follows

$$a_2 = E(y_1^2) \tag{64}$$

$$E(y_1^2) = \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} G(s)\, G(s')\, R(s - s')\, ds \tag{65}$$

where:
G — Green's function,
R — correlation function.

The Green's function must fulfill the equation

$$\frac{d}{dt} G(t - t') - (a_1 - a_2)\, G(t - t') = I\delta(t - t') \tag{66}$$

where:
I — unit matrix,
δ — Dirac's delta function.

For the case considered

$$G(t) = \frac{1}{w_o} I_t e^{-\frac{\varepsilon}{2}(a_2 - 1)} \sin w_o t \tag{67}$$

where:

$$w_o = \left[1 - \frac{1}{4} \varepsilon^2 (a_2 - 1)^2\right]^{1/2}$$

$$I_t = 1, \, t > 0; \; I_t = 0, \, t \leqslant 0$$

In the case when $R_x(t) = \xi^2 \, \delta(t)$ (x — white noise)

$$E(y_1^2) = \xi^2 \frac{1}{2\varepsilon(a_2 - 1)} = a_2$$

$$a_2 = \tfrac{1}{2}\left[1 + \sqrt{1 + \frac{2\xi^2}{\varepsilon}}\right] \tag{68}$$

From (68) it can be seen that if

$$\left(\frac{2\xi^2}{\varepsilon} \to 0\right) = (a^2 - 1). \tag{69}$$

For the system (57) to have periodic solutions on the basis of which a general stationary solution can be constructed the condition (69) must occur.

For
$$R_x(t) = R_o e^{-\beta|t|} \qquad \beta > 0 \tag{70}$$

$$a_2 = 1$$

In further analysis of friction we should also consider theories totally deterministic.

Assuming that ex. in straw madium both the interial and exterial friction can have a mixed character, appropriate rules of friction must consider the dependence of the "dry" and "viscous" components on speed. As it is shown by experiments dry friction decreases, and quite rapidly, with the increase of speed, while viscous friction behaves reversely.

In literature it is possible to meet a model of friction which is based on the following principles

— there is a large difference between the kinetic and static dry friction,

— kinetic dry friction is constant,

— viscous friction increases linearily together with speed.

This model is much simpler than the presented Van der Pol model. However, it has a considerable fault; it does not explain the situation at the beginning of the system at all. Even the assumption of non-linear dependence of the viscous friction coefficient on speed does not help. While in the zero range we observe a considerable value of the friction coefficient (close to the static coefficient) and then either an initial decrease of its value and then an increase, or increase, decrease, and increase again. Exact mathematical explanations made on the basis of physical factors is rather complex and does not seem to be complete asy yet. That is why approaching the problem in the phenomenological way it is possible to approximate the relations quite accurately with differential equations.

Very universal, because of its constants, is the relation

$$T = - c_1 \dot{y}^{c_2} e^{c_3 \dot{y}} . \tag{71}$$

So far in our considerations we have dealt solety with structural theories. Below we present a structural model. It is a rheological model of cereal mass moving in a slot. The aim of these considerations is to turn attention to certain elements of the investigation process which have a strictly methological character.

In a very wide approximation a situation such as the one presented below may occur in the chambers of presses, at a certain place on asangent plane to the screw of the worm in the slot between the edge of the worm and the cover, and in some cases in the slot of the threshing drum.

The plant material in a slot can be described by the position of fibers. If the material consists mainly of straw we can approximate such a medium with the standard model. This model constitutes a combination of the models of Kelvin — Voigt and of Maxwell. Because both the models lead, in respect to damping, to dynamic problems in effect of which we obtain converse results, it is possible to approximate the investigated medium quite well by an appropriate choice of values of component parameters.

The Kelvin — Voigt model is a linear one. The plasticity limit introducing non-linearity into a system does not occur here. The model is good in cases when ccercion is of not too high frequency. Of the medium we describe with this model we must know what is the form of the function of strains $\sigma = \sigma(t)$. This is necessary for solving the basic differential equation.

Then we assume that preliminary deformation is zero. This is a very important remark, since in the case of a working process of whole machine we must assume that on entrance to a successive mechanical system the deformation procured earlier should be treated as "natural".

At a constant strain deformation increases in a continuous way to a determined asymptote. In shock cases deformation increases actualy co fast that already from the initial moment it is possible to assume that it is the same as intended.

Apart from that the body has the property of creep and also the property of elastic delay. But it does not have the ability of relaxation. Apart from that the model has dissipation properties determined by the force of damping in proportion to the deformation rate, which is in accordance with the assumption adopted in equation (17).

The second model is a combination of elasticity and a damper. The model is applicable also in a certain range of frequrncies. A medium of the character of Maxwell's body has the ability of relaxation, which is characterized by therate of strain decrease at a constant deformation. In order to solve the basic differential equation for this model we must know the function of deformation.

The preliminary condition for this equation is the sudden obtaining at moment $t = 0$ a determined deformation. This leads to the establishing that at the initial moment only elastic deformation will occur. The presence of relaxation will cause that the deformation decreases continuously. This model has also a function of creep, but it does not have elastic delay. Both the models become particularly simple if we deal with only formal changes. As to vibrations the two models differ in that in the first one damping is in direct proportion to frequency and in the other in reverse proportion.

In the presentation below we give a basic mathematical description of these models. Basing on data included there we present a probabilistic model of the problem. The problem concernes the finding of the density of distributions of solutions of rheological equations of state. In the K—V model we look for deformations and in the M model for strain function. Both the values are random variables and therefore we must determine the distribution of density of probability describing the changes of their values. If we assume the simplest possible model describing the changes of ε or σ as an effect of cumulation of many causes determined by external coercion and by the structure of the material, we will obtain a certain scheme of random erring. Ex. the state of medium being between two bodies in friction depends on many factors and even if we determined all od them in detail we would not find out what relations occur among them and what values these parameters can adopt.

Let then the sum of effects caused by external and interial factors in relation to the discussed random variables be

$$\underset{(\sigma \wedge \varepsilon)}{S} = x_1 + x_2 + \ldots. + x_n . \tag{72}$$

This sum can be stationary or non-stationary. For the sake of generality of considerations we can assume that the argument of function S does not necessarily denote time.

Let in the range $0 - t$ be n subranges, then the increase

$$S(t)S(0) = \sum_{k=1}^{n} [S(k\tau + \tau) - S(k\tau)] . \tag{73}$$

If we divided ex. a volume of grain into the n parts and investigated their influence of their total resistance to shock (this will be function S), then at $n \rightarrow \infty$ we would obtain an exact picture of the resistance, and not its approximation. Such is the physical sense of the sum of random variables.

For physical purposes it is, however, enough to adopt $n < \infty$. Let us assume for the beginning that x_k are independent. This is not in accordance with the remarks we have made earlier. We treat then the assumption as a hypothesis which can be experimentally check out. This assumption is not necessary, but if it is not fulfilled it greatly complicates the problem on the mathematical side. The fulfillment of the condition of independence causes that this is a Markow's process.

Another restriction

$$x_k = \left| \begin{array}{ll} h & p(x_k = h) = p_1 \\ -h & p(x_k = -h) = p_2 \\ 0 & p(x_k = 0) = p_3 \end{array} \right. \tag{74}$$

where: p — probabilities

$$p_1 + p_2 + p_3 = 1. \tag{75}$$

Thus ex. strain (S) is a result of cummulation of k effects, each of which may with a different probability be h_1—h and 0. This last case indicates a lack of change of state. Since S_σ is realized from state $S_\sigma(0)$ to $S_\sigma(t)$ then the number of all temporarystates should be set for n, and then $\frac{t}{n} = \tau$. Next let us assume that for every argument τ_k the value x_k is realized. The arguments τ_k will then form the string

$$\tau, 2\tau, 3\tau, \ldots \tag{76}$$

with the probabilities p_1, p_2, p_3 respectively.

This then is a case of oscillation of the axis of strains (in the case in question). We should note that for the fulfillment of preliminary conditions the start must take place from the indicated argument. From the probability calculation we know that the argument may be zero for every case, since if it was otherwise, because of the occurence of some strains already then, they could be deducted as a constant value.

The step of ofcillation is then τ and the increase of the value of argument ($\pm h$, 0).

The first task will be the determination of the probability

$$u_{n,k} = p\{S_n = kh\} \tag{78}$$

of the occurence that for the argument $t = n\tau$ the value of the random variable will be $x = kh$. Here $n = 1, 2, \ldots$ and $k = 0, \pm 1, \pm 2, \ldots$ In turn S_{h+1}can adopt the value kh only in the following three mutually exclusive cases

$$\{S_n = (k-1)h \wedge x_{h+1} = h\} \vee \{S_n = (k+1)h \wedge x_{n+1} - h\} \vee \{S_h = kh \wedge x_{h+1} = 0\} \tag{79}$$

$$U_{h+1,k} = p_1 u_{h,k-1} + p_2 u_{n,k+1} + p_3 u_{n,k}. \tag{80}$$

The products in the sum come from the fact that the occurences described by the probabilities p_i and u_n are independent. For $n = 0$

$$\begin{aligned} u_{0,0} &= 1 \qquad k = 0 \\ u_{0,k} &= 1 \qquad k \neq 0 \end{aligned} \tag{81}$$

The differential equation (80) and its prèliminary conditions (81) give the solution of the posed problem. The solution will however be more comprehensible if we pass from the differential equation to a differential equation. In order to do this we must substitute the probabilities by corresponding densities of probability. This may be done if τ is very small. I.e.

$$\lim_{u \to \infty} \left[p\{\nu h < S_n < \mu h\} = u_{n,\nu+1} + u_{n,\nu+2} + \cdots + u_{n,\mu} \right] = \int_{\nu h}^{\mu h} u(t,x)\, dx \tag{82}$$

$$u_{n,k} \cong u(n\tau,kh)h, \; u(n\tau,kh) \simeq \frac{u_n,k}{h}. \tag{83}$$

From the physical point of view the function $u(t,x)$ will be at least twice differentiable.

Equation (80) will take the form

$$u(t+\tau,x) = p_1 u(t,x-h) + p_2 u(t,x+h) + p_3 u(t,x). \tag{84}$$

Taking $p_3 = 1-p_1-p_2$

$$u(t+\tau,x)-u(t,x) = -p_1[u(t,x)-u(t,x-h)] + p_2[u(t,x+h)-u(t,x)].$$

From Taylor's series we get

$$u(t+\tau,x) - u(t,x) = \frac{\partial u(t,x)}{\partial t}\tau + \frac{\partial u(t,x)}{\partial x}\cdot 0 + \frac{1}{2!}\left(\frac{\partial^2 u(t,x)}{\partial t^2}\tau^2 + 2\frac{\partial^2 u(t,x)}{\partial t \partial u}\tau\cdot 0 + \right.$$

$$\left. + \frac{\partial^2 u(t,x)}{\partial x^2}\cdot 0\right) + \cdots = \frac{\partial u(t,x)}{\partial t}\tau + \frac{1}{2}\frac{\partial^2 u(t,x)}{\partial t^2}\tau^2 + \cdots$$

Analogously

$$u(t,x-h)-u(t,x)=-\frac{\partial u(t,x)}{\partial x}h+\frac{1}{2}\frac{\partial u^2(t,x)}{\partial x^2}h^2+\dots$$

$$u(t,x+h)-u(t,x)=\frac{\partial u(t,x)}{\partial x}h+\frac{1}{2}\frac{\partial^2 u(t,x)}{\partial x^2}h^2+\dots$$

Since $\tau^2 \to 0$ then after the transformations we get

$$\frac{\partial u(t,x)}{\partial t}=-\frac{(p_1-p_2)h}{\tau}\cdot\frac{\partial u(t,x)}{\partial x}+\frac{1}{2}\frac{(p_1+p_2)h^2}{\tau}\cdot\frac{\partial^2 u(t,x)}{\partial x^2}+\dots \tag{84.1}$$

where: $\frac{p_1-p_2}{\tau}h^2=a$ — mean transfer on axis for a unit of argument.

Instead of the preliminary conditions (81) we demand now a detailed solution (84) which at $t\to 0$ is convergent to δ Dirac's function, i.e. $u(t,x)\to 0$ for $x\neq 0$ and $u(t,0)\to+\infty$ but in such a way that the integral of the function u is 1 for every $t>0$. At adequate values a and b there is exactly one solution of the equation (84) that has this property and it is precisely this solution that we can adopt as an approximation of the distribution of probability of random variable S_n for large n.

The equation (84) can also be obtained at more general assumptions. Ex.

$$x_k = 0,\ \pm 1,\ \pm 2,\ \pm\dots,\ \pm m. \tag{85}$$

Then a and b will depend on x, i.e. the distribution of probability of every value of strain will depend on the position on the axis of strains. Carrying out a consideration similar to the above we get

$$\frac{\partial u(t,x)}{\partial t}=-\frac{\partial}{\partial x}[a(x)u(t,x)]+\frac{1}{2}\frac{\partial^1}{\partial x^2}[b(x)u(t,x)]. \tag{86}$$

Since strains can adopt only values from a determined range, this fact should be included in the peripheral conditions. Rheological equations of state for the Kelvin—Voigt model and for Maxwell model can be written as

$$\dot{\varepsilon}+\frac{1}{\tau}\varepsilon=\frac{1}{\eta}\sigma(t),\quad \dot{\sigma}+\frac{1}{\tau}\sigma=E\varepsilon(t) \tag{87}$$

where: $\frac{1}{\tau},\frac{1}{\eta},E$ — constants (there is also a way of solving the problem when the constants are not ex. mean values but are random functions),

$\sigma(t)$ and $\varepsilon(t)$ are for the first and second equations respectively the random functions of independent increases alternately serving as coercions.

For the K—V equation and σ for the M equation are also mark processes if certain known conditions are fulfilled.

In order to calculate the coefficient a and b of the equation (84) we should integrate the equations (87) in the ranges $(t, t+\tau)$

$$x(t+\tau)-x(t)=\frac{1}{\eta}\int_{t}^{t+\tau}\sigma(t_1)dt_1-\frac{1}{\tau}\int_{t}^{t+\tau}\varepsilon(t_1)dt_1 \tag{88}$$

$$x(t+\tau)-x(t)=E\int_{t}^{t+\tau}\varepsilon(t_1)dt_1-\frac{1}{\tau}\int_{t}^{t+\tau}\sigma(t_1)\,dt_1. \tag{89}$$

The coercions must have a certain expected value.
Assuming from Table 1 from the peripheral conditions respective σ_0 and ε_0

$$E_{kv}\{[x(t+\tau)-x(t)]\,|\,x=\varepsilon\}=\frac{1}{\tau}(\sigma_o-\varepsilon)\tau'$$

hence

$$a_{kv}=\frac{1}{\tau}(\sigma_o-\varepsilon) \tag{90}$$

(τ — rheological parameter)

$$E_M\left\{[x(t+\tau)-x(t)]\,\middle|\,x=\sigma\right\}=\frac{1}{\tau}(\varepsilon_o-\sigma)\tau'$$

$$a_M=\frac{1}{\tau}(\varepsilon_o-\sigma)$$

$$E_{kv}\left\{[x(t+\tau)-x(t)]^2\,\middle|\,x=\varepsilon\right\}=\frac{1}{\eta^2}\int_{t}^{t+\tau}\int_{t}^{t+\tau}K\sigma(t_2-t_1)\,dt_1dt_2 \tag{91}$$

Assuming for both cases $K(\tau)=\sigma(\tau)$

$$b_{kv}=\frac{1}{\eta^2} \tag{92}$$

$$E_M\{[x(t+\tau)-x(t)]^2 \mid x=\sigma\} = b_M = E^2 \tag{93}$$

where: E — operator.

If the coefficients a and b do not depend on the argument t, then the equation (86) assumes a stationary form

$$\frac{d}{dx}\Big[a(x)\,u(x)\Big] - \frac{1}{2}\frac{d^2}{dx^2}\Big[b(x)\,u(x)\Big] = 0 \tag{94}$$

the solution of which is the function

$$u(x) = \frac{c}{b(x)}\, epx\left\{2\int\limits_{x_o}^{x}\frac{a(x_1)}{b(x_1)}\,dx_1\right\} \tag{95}$$

where: C — the constant of distribution.

For the models in question we will get

$$u_{Kv}(\varepsilon) = c\eta^2\exp\left\{2\int\limits_{\varepsilon_\circ}^{\varepsilon}\frac{\frac{1}{\tau}(\sigma_\circ - \varepsilon_1)}{\frac{1}{\eta^2}}\,d\varepsilon_1\right\}.$$

Since $\varepsilon_0 = 0$

$$u_{Kv}(\varepsilon) = c\eta^2\exp\left\{\frac{\eta^2}{\tau}(2\sigma_\circ\,\varepsilon - \varepsilon^2)\right\}, \tag{96}$$

$$u_M(5) = \frac{c}{E^2}\exp\left\{2\int\limits_{\sigma_\circ}^{\sigma}\frac{\frac{1}{\tau}(\varepsilon_\circ - \sigma_1)}{E^2}\,d\sigma_1\right\}.$$

Since $\sigma_0 = E\varepsilon_0$

$$u_M(\sigma) = \frac{c}{E^2}\exp\left\{\frac{\varepsilon_\circ{}^2(E-2)}{\tau E}\right\}\exp\left\{\frac{1}{\tau E^2}(2\varepsilon_\circ\sigma - \sigma^2)\right\}. \tag{97}$$

A helpfull thing in this respect will be the determination of the mean $\bar{x}$ and the variation $\sigma_x{}^2$ for ε and σ. Explicit solution can be obtained in the case when a and b are linear functions ε and σ, i.s.

$$a(t,\varepsilon) = a + a_1\varepsilon,\; a(t,\sigma) = a'_\circ + a'_1\sigma \tag{98}$$

$$b(t,\varepsilon) = \beta_\circ + \beta_1\varepsilon,\; b(t,\sigma) = \beta'_\circ + \beta'_1\sigma. \tag{99}$$

Applying the apparatus of characteristic functions and elementary equations of first degree we obtain

$$\bar{\varepsilon}(t) = \frac{a_\circ}{a_1}(e^{a_1 t} - 1) + \varepsilon c^{a_1 t},$$

$$\bar{\sigma}(t) = \frac{a'_\circ}{2a'_1}(e^{2a'_1 t} - 1) + \sigma e^{a'_1 t} \quad (100)$$

$$\sigma^2_\varepsilon(t) = \frac{\beta_\circ}{2a_1}(e^{2a_1 t} - 1),$$

$$\sigma^2_\sigma(t) = \frac{\beta'_\circ}{2a'_1}(e^{2a'_1 t} - 1). \quad (101)$$

For the two rheological models in question

$$a_\circ = \frac{1}{\tau}\sigma_\circ, \; a_1 = -\frac{1}{\tau}, \; a'_\circ = \frac{1}{\tau}\varepsilon_\circ, \; a'_1 = -\frac{1}{\tau}$$

$$\beta_\circ = \frac{1}{\eta^2}, \; \beta_1 = 0, \; \beta'_\circ = E^2, \; \beta'_1 = 0. \quad (102)$$

Hence

$$\bar{\varepsilon}(t) = -\sigma_\circ(e^{-\frac{t}{\tau}} - 1) + \varepsilon e^{-\frac{t}{\tau}},$$

$$\bar{\sigma}(t) = -\varepsilon_\circ(e^{-\frac{t}{\tau}} - 1) + \sigma e^{-\frac{t}{\tau}},$$

$$\sigma^2_\varepsilon(t) = -\frac{\tau}{2\eta^2}(e^{\frac{2t}{\tau}} - 1),$$

$$\sigma^2_\sigma(t) = -\frac{E^2\tau}{2}(e^{-2\frac{t}{\tau}} - 1).$$

On the basis of the last relations we can determine the constants in the distributions (96) and (97)

$$c_{\eta^2} = \frac{1}{\sqrt{\pi\frac{\tau}{\eta^2}(1 - \exp[-2t\tau^{-1}])}}$$

$$c_{Kv} = \frac{1}{\sqrt{\pi\tau\eta^2(1 - \exp[-2t\tau^{-1}])}}. \quad (103)$$

$$\frac{e}{E^2} = \frac{1}{\sqrt{\pi E^2\tau(1 - \exp[-2t\tau^{-1})]}},$$

$$c_M = \frac{E}{\sqrt{\pi\tau(1-\exp[-2t\tau^{-1}])}}. \qquad (104)$$

If we worked out similar calculation schemes for all the commonly known rheological models then we could, on the basis of experimental data, choose the most probable structural model.

Making measurements it is possible to determine the parameters of their distributions and in the case of normality to compare them with those calculated theoretically. Theoretical calculations similar to those presented above should be done for all commonly used structural models.

As an appropriate model for a given medium we should treat that which gives the greatest from the point of probability agreement between theoretical calculations and experimental data. The agreement could be investigated with the help of standard tests.

Coming back now to the example of the rheological model of straw mass it seems to the point, because of the contents of other plants and other organic elements such as broken and often crumbled leaves, to treat the medium as two-phase and to add in series Maxwell body to the standard model. Such a construction of the model would be argued for by the fact that the second phase will have a much higher elasticity and that only damping wille be of any greater influence on the whole medium. Of course equations will be here much more complex.

Recapitulating we should state that presented problem of applying stochastic processes in the finding of a rheological models is important because of the probabilistic characteristics of the medium.

The finding of the model with deterministic methods is in practice rather limited because of ex. great distribution of constant values, the measure of which is not in this case considered. The present paper is an attempt at indicating this problem and its main aim is to initiate discussion of it.

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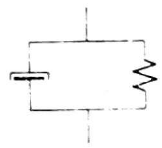
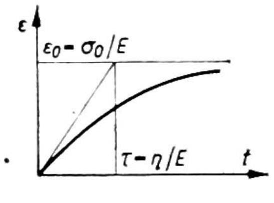
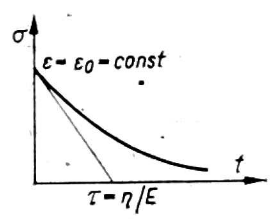
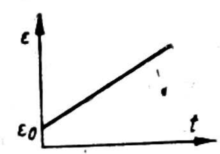
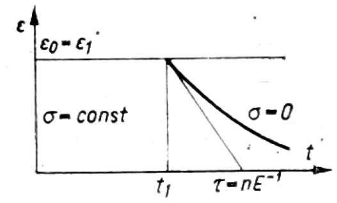
	Kelvin-Voigt model		Maxwell's model	
Damping energy ξ_T Frequency ω	$\xi_T \overset{df}{=} a\,\omega$		$\xi_T \overset{df}{=} \beta \frac{1}{\omega}$	
Relation between strains and deformations	$\varepsilon_s = \varepsilon_l = \varepsilon,$ s — elastic part,	$\sigma_s + \sigma_l = \sigma$ l — viscous part	$\varepsilon_s + \varepsilon_l = \varepsilon,$ $\sigma_s = E\,\varepsilon,$	$\sigma_s = \sigma_l = \sigma$ $\sigma_l = \eta \cdot \dot{\varepsilon}$
Assumptions	$\sigma = \sigma(t)$		$\varepsilon = \varepsilon(t)$	
Rheological equation of state	$\sigma = E\,\varepsilon + \eta\,\dot{\varepsilon} = E(\varepsilon + \tau\,\dot{\varepsilon})$ $\dot{\varepsilon} = \frac{\sigma(t)}{\eta} - \frac{\varepsilon}{\tau}$		$\dot{\varepsilon} = \frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta},$ $\dot{\sigma} = E\,\dot{\varepsilon}(t) - \frac{\sigma}{\tau}$	$\sigma = \eta\,\dot{\varepsilon} - \tau\,\dot{\sigma}$
E — modulus of elasticity η — coefficient of viscosity	$\tau = \frac{\eta}{E}$ — time of elastic delay characterized by speed of deformation decreas after removal of loading		$\tau = \frac{\eta}{E}$ — time of relaxation, characterizing the speed of strain decrease at constant deformation	
Function sought	$\varepsilon = \varepsilon(t)$		$\sigma = \sigma(t)$	
Boundary conditions	$\varepsilon_o = \varepsilon(o) = 0,$	$\sigma(t) = \sigma_o = \text{const}$	$\sigma_o = \sigma(o) = E\,\varepsilon_o,$	$\varepsilon(t) = \varepsilon_o = \text{const}$
Solution	$\varepsilon(t) = \frac{\sigma_o}{E}\left(1 - e^{-\frac{t}{\tau}}\right)$ 		$\sigma(t) = \sigma_o e^{-\frac{t}{\tau}}$ 	
Function of creep	$\varphi(t) = \frac{\varepsilon(t)}{\sigma} = \frac{1}{E}\left(1 - e^{-\frac{t}{\tau}}\right)$		$t = 0, \quad \sigma(t) = \sigma_o = \text{const},$ $\varepsilon(t) = \varepsilon_o\left(1 + \frac{t}{\tau}\right)$ $\varphi(t) = \frac{\varepsilon(t)}{\sigma_o} = \frac{1}{E}\left(1 + \frac{t}{\tau}\right)$ 	

cd. table

	Kelvin-Voigt model	Maxwell's model
Function of elastic delay	$\varepsilon(t) = \frac{\sigma_0}{E}\left(e^{-\frac{t_1}{\tau}}\right) e^{-\frac{t}{\tau}} = \varepsilon_1 e^{-\frac{t}{\tau}}$ 	None
Funation of rel ıxation	None	$\psi(t) = \frac{\sigma(t)}{\varepsilon_0} = E_0 e^{-\frac{t}{\tau}}$

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PROBABILISTYCZNE ASPEKTY MODELI REOLOGICZNYCH

Streszczenie

Generalnym założeniem pracy jest hipoteza, że nie istnieje możliwość napisania uogólnionego równania konstytutywnego dla ciała scharakteryzowanego sprężystością, lepkością i plastycznością o lokalnych rozkładach statystycznych i różnorodnych złożonych kombinacjach, a więc takich, w których model reologiczny jest funkcją położenia i zależy od nałożonych na ruch warunków brzegowych.

Poszukiwanie modelu reologicznego jest sprawą konieczną z punktu widzenia projektanta urządzeń, dla których ośrodkiem powtarzanym jest właśnie materiał roślinny.

Jest to sprawa o tyle istotna, że nie można mówić o optymalizacji urządzenia, jeżeli nie zna się rozkładów rozwiązań konstytutywnych medium.

Poszukiwanie modelu reologicznego ośrodka pracującego w danym układzie mechanicznym można realizować wieloma drogami, jednakże wydaje się, że metody stochastyczne są tutaj najbardziej właściwe.

Metoda stochastyczna w tym przypadku zapewnia znalezienie dla danych warunków najbardziej prawdopodobnego modelu. Założeniem jej jest poszukiwanie rozkładu połączeń: szeregowych, równoległych oraz szeregowo-równoległych poszczególnych członów składowych modelu. Poszukiwanie to mogłoby być realizowane na drodze błądzenia po karcie, której węzłami są człony reologiczne, a przejścia są zmiennymi będącymi szumami.

$$Z = U + V$$

gdzie: U — jest członem deterministycznym, V — członem losowym.

Rozkłady rozwiązań konstytutywnych uzyskiwać można z parabolicznego równania typu równania Kołmogorowa, którego postać uzyskuje się z procesu błądzenia.

Wstępne informacje o procesie wchodzące w wektor U są natury fizycznej i można je uzyskać na drodze bezpośrednich pomiarów wytrzymałościowych.

Problematyka przedstawoina w pracy ma charakter poszukiwań i nie stanowi uniwersalnej metody dla wszystkich tego typu problemów.

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ПРОБАБИЛИСТИЧЕСКИЕ АСПЕКТЫ МОДЕЛИ РЕОЛОГИЧЕСКИХ

Резюме

Генеральной предпосылкой работы является гипотеза, что не существует возможность написания обобщенного уравнения, конститутивного для тела, охарактеризованного упругостью, липкостью и пластичностью с местными статистическими распределениями и различными взаимокомбинациями, значит, такими, в которых реологическая модель является функцией положения и зависит от наложенных на движение береговых условий.

Поиски реологической модели являются проблемой, необходимой с точки зрения проектирования приборов, для которых повторяемой средой является именно растительный материал.

Это — настолько существенная проблема, что нельза говорить об оптимализации прибора, если не известны распределения решений конститутивных уравнений медиума.

Поиски реологической модели среды, работающей в данной механической системе, могут осуществляться многими путями, однако, кажется, что стохастические методы являются тут наиболее подходящими.

Стохастический метод в этом случае позволяет найти наиболее вероятную модель для данных условий. Предпосылкой для нее являются поиски распределения последовательных, параллельных и последовательно-параллельных соединений отдельных составляющих членов модели. Эти поиски могли б осуществляться путем блуждания по решетке, узлы которой являются реологическими членами, а переходы — переменными, являющимися шумами.

$$Z = U + V$$

где: U — детерминистический член, V — случайный член.

Распределения конститувных решений можно получать из параболического уравнения типа уравнения Колмогорова, форму каторого находят в процессе блуждания.