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M-BETTER TYPE S BLOCK DESIGN FOR RESEARCH INTO ALTERNATIVE METHODS OF PLANT PROTECTION

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Summary

The specific character of research into plant protection entails a need for studies on the planning and analysis of such experiments. We present here the properties of certain block designs. Type S block designs are considered for experiments in which there are v levels of an experimental factor in addition to one control treatment.

We consider the case in which optimality can be achieved relative to the Loewner ordering among information matrices. We prove that within the class of Type S block designs, we can compare Fisher's information matrices in the Loewner ordering. This fact enables the application of the theory of M-optimality or the theory of choice of an M-better design. We present these considerations in response to questions from plant protection researchers.

Keywords and phrases: M-better design, type S block design

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1. Introduction

Block designs for experiments with v test treatments and one control treatment were first considered in Hoblyn et al (1954). Statistical properties of such designs have been discussed by many authors, but the supplemented balance block design (Type S design) was defined by Pearce (1960). The properties of the supplemented balance block design were described in Hedayat et al (1988). Construction methods for Type S block designs were introduced and described in Gupta and Kageyama (1993). A table is given with parameters of many such block designs for $4 \le v \le 24$. The supplemented balance block design has been considered in recent years by, for example, Hinkelmann and Kempthorne (2005) and Bailey (2008), and earlier by Calinski and Ceranka (1974), Kozłowska and Błażczak (1990) and Pearce (1995).

Type S block designs are used for plant protection experiments because these frequently compare a few test treatments with one control treatment. Many such experiments have to be limited in terms of the numbers of replications of test treatments. In this situation, Type S block designs with a small number of replications for test treatments and a greater number for the control are used. These are also connected block designs.

In this paper we describe the special problem of the choice of a Type S block design in response to questions from plant protection researchers.

2. Framework

In this section we consider only connected block designs. We have v test treatments and one control treatment added. These treatments are arranged on experimental units which are grouped in b blocks in a such way that Fisher's information matrix has the following form:

$$\mathbf{C} = \begin{bmatrix} vw_0 & -w_0\mathbf{1}' \\ -w_0\mathbf{1} & (w_0 + vw_1)\mathbf{I} - w_1\mathbf{11}' \end{bmatrix}, \tag{2.1}$$

where w_0 (> 0) and w_1 (> 0) denote some known parameters, $\mathbf{I}$ is the unit matrix of order v and $\mathbf{1}$ is the v - dimensional vector of ones. Gupta and Kageyama (1993) give a definition according to which a design is said to be Type S if $\text{var}(\hat{\tau}_i - \hat{\tau}_0) < \text{var}(\hat{\tau}_i - \hat{\tau}_j)$ for $i \ne j = 1,2,...,v$, where $\hat{\tau}_k$ denotes estimator of parameter τ_k (k = 0, 1, …, v) and $\boldsymbol{\tau} = (\tau_0, \tau_1, \tau_2, ..., \tau_v)'$ is the vector of treatment parameters for the control and test treatments respectively. Since the information matrix has the form (2.1) and $\text{var}(\hat{\tau}_i - \hat{\tau}_0) = (w_0 + w_1)(w_0 + vw_1)^{-1} w_0^{-1} \sigma^2$, $\text{var}(\hat{\tau}_i - \hat{\tau}_j) = 2(w_0 + vw_1)^{-1} \sigma^2$ for $i \ne j = 1,2,...,v$, hence it is easy to see that $w_1 < w_0$.

For comparing Type S block designs we apply the theory described by Bagchi and Bagchi (2001). The authors state that a design $d_2 \in D$ is said to be better than another design $d_1 \in D$ in the sense of majorization (M-better) if the vector γ_{d_2} of eigenvalues of the information matrix for design d_2 is weakly majorized by the vector γ_{d_1} of eigenvalues of the information matrix for design d_1. Bagchi and Bagchi state that a design $d_0 \in D$ is said to be optimal in the sense of majorization (M-optimal) if it is M-better than every other design in *D*. Hence we recall the theory described by Marshall et al (2011, p. 8) and Pukelsheim (2006). For any row vector $\mathbf{x} = (x_1, x_2, \ldots, x_v)' \in \mathrm{R}^v$ for which $x_1 \le x_2 \le \ldots \le x_v$ we can say that the vector $\mathbf{x}$ is majorized by the vector $\mathbf{y}$, denoted as $\mathbf{x} \prec \mathbf{y}$, if the following relations hold

$$\sum_{i=1}^{k} x_i \ge \sum_{i=1}^{k} y_i \text{ , and } \sum_{i=1}^{v} x_i = \sum_{i=1}^{v} y_i \text{ for } k = 1, 2, \ldots, v-1.$$

We can say too that the vector $\mathbf{x}$ is weakly majorized by the vector $\mathbf{y}$, denoted as $\mathbf{x} \prec^{w} \mathbf{y}$, if for $k = 1, 2, \ldots, v-1$ the first of the above relations holds.

3. Background

Let $\mathbf{A}$ be any $(v+1)\times(v+1)$ symmetric positive semidefinite matrix. It is known that all its eigenvalues are non-negative, its determinant is equal to zero and it is a singular matrix. Other properties require some results that will be proved now.

Let matrix $\mathbf{A}$ be partitioned as follows:

$$\mathbf{A} = \begin{bmatrix} a\mathbf{I} + b\mathbf{11}' & -c\mathbf{1} \\ -c\mathbf{1}' & d \end{bmatrix}, \tag{3.1}$$

where $a\mathbf{I} + b\mathbf{11}'$ is a $v \times v$ symmetric positive definite submatrix. Hence all eigenvalues of the matrix $a\mathbf{I} + b\mathbf{11}'$ are positive and have the form $b\varepsilon_i + a$, where ε_i denotes the eigenvalues of the square $v \times v$ matrix $\mathbf{11}'$. For $i = 1$ we have $\varepsilon_1 = v$ and for $i = 2, \ldots, v$ we have $\varepsilon_i = 0$. Because the eigenvalues of a positive definite matrix are positive, hence $bv + a > 0$ and $a > 0$. On the other hand when the inequalities $bv + a > 0$ and $a > 0$ hold, the matrix $a\mathbf{I} + b\mathbf{11}'$ is positive definite, hence we have

Lemma 3.1. A necessary and sufficient condition for the $v \times v$ matrix $a\mathbf{I} + b\mathbf{11}'$ to be positive definite is the truth of the inequalities $bv + a > 0$ and $a > 0$.

In this case the determinant of the matrix $a\mathbf{I} + b\mathbf{11}'$ has the form $\det(a\mathbf{I} + b\mathbf{11}') = a^{v-1}(bv + a)$. Because we assume that the matrix $a\mathbf{I} + b\mathbf{11}'$ is positive definite, hence $\det(a\mathbf{I} + b\mathbf{11}') > 0$. For any positive definite matrix Sylvester's theorem holds (see for example Gilbert, 1991). It is known when a matrix is positive definite, all of the leading principal minors are positive. Hence for the first principal minor we have $a + b > 0$.

Now we calculate the determinant of the matrix **A**. Applying formula (1.1) from Powell (2011), we obtain

$$\det(\mathbf{A}) = \det\begin{bmatrix} a\mathbf{I} + b\mathbf{11}' & -c\mathbf{1} \\ -c\mathbf{1}' & d \end{bmatrix} = d\det(a\mathbf{I} + (b - d^{-1}c^2)\mathbf{11}') =$$
$$= da^{v-1}((b - d^{-1}c^2)v + a) = a^{v-1}(-c^2v + d(bv + a)).$$

If $\det(\mathbf{A}) = 0$ then $a^{v-1}(-c^2v + d(bv + a)) = 0 \Leftrightarrow d = \dfrac{c^2v}{bv + a}$. Hence we have

Lemma 3.2. Let **A** be a square matrix of the form (3.1), where $bv + a > 0$ and $a > 0$. Then **A** is positive semidefinite if and only if the diagonal element d equals $c^2v(bv + a)^{-1}$.

Using the properties of determinants, we can see that:

− for any symmetric matrix **A** of the form (3.1)

$$\det\left(\begin{bmatrix} a\mathbf{I} + b\mathbf{11}' & -c\mathbf{1} \\ -c\mathbf{1}' & d \end{bmatrix}\right) = \det\left(\begin{bmatrix} d & -c\mathbf{1}' \\ -c\mathbf{1} & a\mathbf{I} + b\mathbf{11}' \end{bmatrix}\right);$$

− any square partitioned matrix $\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}$ is positive semidefinite if and only if so is the matrix $\begin{bmatrix} \mathbf{D} & \mathbf{C} \\ \mathbf{B} & \mathbf{A} \end{bmatrix}$.

4. M-better type S block designs

Choice of an appropriate experimental plan allows one to obtain the maximum amount of information, in a certain sense, which is contained in Fisher's information matrix. **A** decision concerning the choice of a better experimental plan is obtained by comparing the information matrices. A full comparison of information matrices is known as the Loewner ordering of symmetric matrices.

Recall that according to the Loewner ordering (Marshall et al. 2011, p. 670), for two Hermitian matrices $\mathbf{A}$ and $\mathbf{B}$, $\mathbf{A} \le \mathbf{B}$ means $\mathbf{B} - \mathbf{A}$ is positive semidefinite, $\mathbf{A} < \mathbf{B}$ means $\mathbf{B} - \mathbf{A}$ is positive definite.

It is known that for two matrices $\mathbf{C}_1$ and $\mathbf{C}_2$ which are in the Loewner ordering ($\mathbf{C}_2 \ge \mathbf{C}_1$), vector of eigenvalues of matrix $\mathbf{C}_2$, $\boldsymbol{\gamma}_2$ is weakly majorized by the vector of eigenvalues of matrix $\mathbf{C}_1$, $\boldsymbol{\gamma}_1$ ($\boldsymbol{\gamma}_2 \prec^w \boldsymbol{\gamma}_1$) (Marshall et al. 2011, p. 360).

Let $\mathbf{C}_1$ and $\mathbf{C}_2$ denote the matrices of the form (2.1). In this case $\mathbf{C}_2 - \mathbf{C}_1$ has the following form:

$$\mathbf{C}_2 - \mathbf{C}_1 = \begin{bmatrix} v(w_{20} - w_{10}) & -(w_{20} - w_{10})\mathbf{1}' \\ -(w_{20} - w_{10})\mathbf{1} & ((w_{20} - w_{10}) + v(w_{21} - w_{11}))\mathbf{I} - (w_{21} - w_{11})\mathbf{11}' \end{bmatrix}. \quad (4.1)$$

Theorem 4.1. For two Type S designs d_1 and d_2 from class $D(v+1)$, their information matrices are in the Loewner ordering ($\mathbf{C}_2 \ge \mathbf{C}_1$) if the inequalities $(w_{20} - w_{10}) \ge 0$ and $(w_{21} - w_{11}) \ge 0$ hold.

Proof: Consider the four possibilities. Firstly, if $(w_{20} - w_{10}) = 0$ and $(w_{21} - w_{11}) = 0$ then $\mathbf{C}_2 - \mathbf{C}_1$ is the zero matrix, and thus is semipositive definite.

Secondly, if $(w_{20} - w_{10}) = 0$ and $(w_{21} - w_{11}) > 0$ then

$$\mathbf{C}_2 - \mathbf{C}_1 = \begin{bmatrix} 0 & \mathbf{0}' \\ \mathbf{0} & v(w_{21} - w_{11})\mathbf{I} - (w_{21} - w_{11})\mathbf{11}' \end{bmatrix}.$$

In fact, the distinct eigenvalues of $\mathbf{C}_2 - \mathbf{C}_1$ coincide with the eigenvalues of $v(w_{21} - w_{11})\mathbf{I} - (w_{21} - w_{11})\mathbf{11}'$ which in turn are equal to $v\,(w_{21} - w_{11})$ and 0. Hence $\mathbf{C}_2 - \mathbf{C}_1$ is semipositive definite.

Thirdly, if $(w_{20} - w_{10}) > 0$ and $(w_{21} - w_{11}) = 0$ then

$$\mathbf{C}_2 - \mathbf{C}_1 = \begin{bmatrix} v(w_{20} - w_{10}) & -(w_{20} - w_{10})\mathbf{1}' \\ -(w_{20} - w_{10})\mathbf{1} & (w_{20} - w_{10})\mathbf{I} \end{bmatrix}.$$

It is easy to see that $(\mathbf{C}_2 - \mathbf{C}_1)\mathbf{1} = 0$. Thus that the matrix $\mathbf{C}_2 - \mathbf{C}_1$ is semipositive definite.

Fourthly, the determinant of a matrix $\mathbf{C}_2 - \mathbf{C}_1$ of the form (4.1) is equal to the determinant of the following matrix

$$\begin{bmatrix} ((w_{20} - w_{10}) + v(w_{21} - w_{11}))\mathbf{I} - (w_{21} - w_{11})\mathbf{11}' & -(w_{20} - w_{10})\mathbf{1} \\ -(w_{20} - w_{10})\mathbf{1}' & v(w_{20} - w_{10}) \end{bmatrix}.$$

If $(w_{20} - w_{10}) > 0$ and $(w_{21} - w_{11}) > 0$ then from Lemma 3.1 we obtain that the matrix $((w_{20} - w_{10}) + v(w_{21} - w_{11}))\mathbf{I} - (w_{21} - w_{11})\mathbf{11}'$ is positive definite. Since the last diagonal element is equal to $v(w_{20} - w_{10})$, from lemma 3.2 the matrix $\mathbf{C}_2 - \mathbf{C}_1$ is semipositive definite. Hence the proof is complete. □

Proposition 4.1. A block design d_2 is M-better than d_1 in the class of type S block designs $D(v+1)$ if $w_{20} \geq w_{10}$ and $w_{21} \geq w_{11}$.

5. Research on alternative methods of plant protection

As a result of intensive changes in plant production and of environmental changes in agrocenoses, certain agrophages have been causing increasing amounts of damage to agricultural crops. As a result of a European Parliament and Council Directive adopted on 13 January 2009, the countries of Europe have been required to implement integrated plant protection. One of the principles of this protection is the implementation of strategies which minimize the use of chemical pesticides. It therefore becomes necessary to seek environmentally safe substances and to develop non-chemical methods for protecting plants. Studies have been made of the effectiveness of iron phosphate and *P. hermaphrodita* nematode doses in limiting damage to Chinese cabbage seedlings in combating Polish populations of the invasive slug species *Arion lusitanicus*. Four test treatments and one control treatment were used, and the experiment was carried out in a Type S block design. We considered different schemes of arrangement of treatments on experimental units, namely Type S block designs from the class $D(v+1=5)$. In this class there are type S block designs with the following parameters:

1. $\nu = 4, r_1 = 11, r_0 = 12, k_1 = 2, k_2 = 3, b_1 = 10, b_2 = 12, w_1 = 1.5, w_0 = 2;$

2. $\nu = 4, r_1 = 11, r_0 = 18, k_1 = 2, k_2 = 3, b_1 = 4, b_2 = 18, w_1 = 1.33, w_0 = 3;$

3. $\nu = 4, r_1 = 13, r_0 = 12, k_1 = 2, k_2 = 3, b_1 = 14, b_2 = 12, w_1 = 1.83, w_0 = 2;$

4. $\nu = 4, r_1 = 13, r_0 = 18, k_1 = 2, k_2 = 3, b_1 = 8, b_2 = 18, w_1 = 1.67, w_0 = 3;$

5. $\nu = 4, r_1 = 16, r_0 = 18, k_1 = 2, k_2 = 3, b_1 = 14, b_2 = 18, w_1 = 2.17, w_0 = 3;$

6. $\nu = 4, r_1 = 17, r_0 = 18, k_1 = 2, k_2 = 3, b_1 = 16, b_2 = 18, w_1 = 2.33, w_0 = 3;$

where v denotes number of test treatments, r_1 – number of test treatments replications, r_0 – number of control treatment replications, b_1 - number of blocks with a capacity of k_1, b_2 - number of blocks with a capacity of k_2, w_1 and w_0 are the same as in (2.1). These parameters satisfy the equation $vr_1 + r_0 = b_1k_1 + b_2k_2$.

The last design is M-better than others with the parameters given above. There are four treatments with seventeen replications and one control treatment with eighteen replications which are arranged on 86 experimental units. These units are grouped in sixteen blocks of size two and next eighteen blocks of size three. The design is given by Gupta and Kageyama (1993), described as 4SR1 + R5S + 2(2, 2)S, according to Clatworthy (1973) notation. The incidence matrix of the design may be written in the following form

$$\mathbf{N} = \begin{bmatrix} \mathbf{0}_4^{'} & \mathbf{0}_4^{'} & \mathbf{0}_4^{'} & \mathbf{0}_4^{'} & 1 & 1 & 1 & 1 & \mathbf{1}_7^{'} & \mathbf{1}_7^{'} \\ \mathbf{1}_4^{'} & \mathbf{0}_4^{'} & \mathbf{1}_4^{'} & \mathbf{0}_4^{'} & 1 & 0 & 1 & 0 & \mathbf{1}_7^{'} & \mathbf{0}_7^{'} \\ \mathbf{1}_4^{'} & \mathbf{0}_4^{'} & \mathbf{0}_4^{'} & \mathbf{1}_4^{'} & 1 & 0 & 0 & 1 & \mathbf{0}_7^{'} & \mathbf{1}_7^{'} \\ \mathbf{0}_4^{'} & \mathbf{1}_4^{'} & \mathbf{0}_4^{'} & \mathbf{1}_4^{'} & 0 & 1 & 0 & 1 & \mathbf{1}_7^{'} & \mathbf{0}_7^{'} \\ \mathbf{0}_4^{'} & \mathbf{1}_4^{'} & \mathbf{1}_4^{'} & \mathbf{0}_4^{'} & 0 & 1 & 1 & 0 & \mathbf{0}_7^{'} & \mathbf{1}_7^{'} \end{bmatrix},$$

where $\mathbf{0}_v$ and $\mathbf{1}_v$ are the v - dimensional vectors of zeros and ones, respectively.

The experiment was performed under laboratory conditions. Every two days, damage to the plants was determined by using a five-point scale (damage to 0, 25, 50, 75 and 100% of the plant surface). The result showed that *P. hermaphrodita* reduced damage to cabbage plants on the seventh day of observations. Ferramol significantly decreased the damage to plants from the first day. The research indicated that both products are a valuable alternative to currently used molluscicides.

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