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Hamiltonian cycle containg selected sets of edges of a graph

Grzegorz Gancarzewicz

Institute of Mathematics, Tadeusz Kościuszko Cracow University of Technology, 24 Warszawska Str., 31-155 Cracow, Poland

E-mail address: gancarz@pk.edu.pl

ABSTRACT

The aim of this paper is to characterize for every $k \geq 1$ all $(l + 3)$-connected graphs G on $n \geq 3$ vertices satisfying $P(n + k)$: $\mathrm{d}_G(x,y) = 2 \ \Rightarrow \ \max\{\mathrm{d}(x,G), \mathrm{d}(y,G)\} \geq \dfrac{n+k}{2}$ for each pair of vertices x and y in G, such that there is a path system S of length k with l internal vertices which components are paths of length at most 2 satisfying:

$$P: \ u_1u_2u_3 \subset S \text{ and } \mathrm{d}(u_1,G), \ \mathrm{d}(u_2,G) \geq \frac{n+k}{2} \ \Rightarrow \ \mathrm{d}(u_3,G) \geq \frac{n+k}{2},$$

such that S is not contained in any hamiltonian cycle of G.

***Keywords*:** Cycle; hamiltonian cycle; matching; path

1. INTRODUCTION

We consider only finite graphs without loops and multiple edges. By V or V(G) we denote the vertex set of graph G and respectively by E or E(G) the edge set of G. By d(x,G) or d(x) we denote *the degree of a vertex x in the graph G* and by d(x,y) or dG(x,y) *the distance between x and y in G.*

Definition 1.1. (cf [7]) *Let* $k, s_1, \ldots s_l$ *be positive integers. We call S a path system of length k if the connected components of S are paths:*

$$P^1 : \qquad x_0^1 x_1^1 \ldots x_{s_1}^1,$$
$$\vdots$$
$$P^l : \qquad x_0^l x_1^l \ldots x_{s_l}^l$$

And $\sum_{i=1}^{l} s_i = k$.

Let S be a path system of length k and let $x \in V(S)$. We shall call x *an internal vertex* if x is an internal vertex (cf [2]) in one of the paths $P^1, \ldots, P^l$.

If q denotes the number of internal vertices in a path system S of length k then $0 \leq q \leq k - 1$. If $q = 0$ then S is *a k-matching* (i.e. a set of k independent edges).

Let G be a graph and let S be a path system of length k in G. Let paths $P^1 : x_0^1 x_1^1 \ldots x_{s1}^1$, …, $P^l : x_0^l x_1^l \ldots x_{sl}^l$ be components of S. We can define a new graph $\tilde{G}$ and a matching M_S in

$$M_S = \{xy : x = x_0^i, y = x_{s_i}^i, i = 1, \ldots, l\}$$

$$V(\widetilde{G}) = V(G) \setminus \bigcup_{i=1}^{l} \{x_1^i, \ldots, x_{s_i-1}^i\}$$

$$E(\widetilde{G}) = M_S \cup \{xy : x, y \in V(\widetilde{G}) \text{ and } xy \in E(G)\}$$

Let H be a subgraph or a matching of G. By $G \backslash H$ we denote the graph obtained from G by the deletion of the edges of H.

Definition 1.2. *F is an H-edge cut-set of G if and only if* $F \subset E(H)$ *and* $G \backslash F$ *is not connected.*

Definition 1.3. *F is said to be a minimal H-edge cut-set of G if and only if F is an H-edge cut-set of G which has no proper subset being an edge cut-set of G.*

Definition 1.4. (cf [5]) *Let G be a graph on* $n \geq 3$ *vertices and* $k \geq 0$. *G is k-edge-hamiltonian if for every path system P of length at most k there exists a hamiltonian cycle of G containing P.*

Let G be a graph and $H \subset G$ a subgraph of G. For a vertex $x \in V(G)$ we define the set $N_H(x) = \{y \in V(H) : xy \in E(G)\}$. Let H and D be two subgraphs of G. $E(D,H) = \{xy \in E(G) : x \in V(D)$ and $y \in V(H)\}$. For a set of vertices A of a graph G we define the graph $G(A)$ as the subgraph induced in G by A.

In the proof we will only use oriented cycles and paths. Let C be a cycle and $x \in V(C)$, then x^- is *the predecessor of* x and x^+ is its *successor*. We denote the number of components of a graph G by $\omega(G)$.

Definition 1.5. (cf [1]) *Let W be a property defined for all graphs of order n and let k be a non-negative integer. The property W is said to be k-stable if whenever G + xy has property W and* $d(x,G) + d(y,G) \geq k$ *then G itself has property W.*
J.A. Bondy and V. Chvátal [1] proved the following theorem, which we shall need in the proof of our main result:

Theorem 1.1. *Let n and k be positive integers with* $k \leq n - 3$. *Then the property of being k-edge-hamiltonian is* $(n + k)$*-stable.*

In 1960 O. Ore [6] proved the following:

Theorem 1.2. *Let G be a graph on* $n \geq 3$ *vertices. If for all nonadjacent vertices* $x,y \in V(G)$ *we have*

$$d(x, G) + d(y, G) \geq n$$

then G is hamiltonian.

Geng-Hua Fan [3] has shown:

Theorem 1.3. *Let G be a 2-connected graph on* $n \geq 3$ *vertices. If G satisfies*

$$P(n): \quad d_G(x, y) = 2 \Rightarrow \max\{d(x, G), d(y, G)\} \geq \frac{n}{2}$$

for each pair of vertices x and y in G, then G is hamiltonian.

The condition for degree sum in Theorem 1.2 is called *an Ore condition* or *an Ore type condition for graph G* and the condition $P(k)$ is called *a Fan condition* or *a Fan type condition for graph G.*

Later many *Fan type theorems* and *Ore type theorems* has been shown.

Now we shall present Las Vargnas [8] condition $\mathcal{L}_{n,s}$.

Definition 1.6. *Let G be graph on* $n \geq 2$ *vertices and let s be an integer such that* $0 \leq s \leq n$. *G satisfies Las Vargnas condition* $\mathcal{L}_{n,s}$ *if there is an arrangement* $x_1, \ldots, x_n$ *of vertices of G such that for all i,j if*

$$1 \leq i < j \leq n, \ \ i + j \geq n - s, \ \ x_i x_j \notin E(G),$$

$$d(x_i, G) \leq i + s \ \ \textit{and} \ d(x_j, G) \leq j + s - 1$$

then $d(x_i,G) + d(x_j,G) \geq n + s$.

Las Vargnas [8] proved the following theorem:

Theorem 1.4. *Let G be a graph on* $n \geq 3$ *vertices and let* $0 \leq s \leq n - 1$. *If G satisfies* $\mathcal{L}_{n,s}$ *then G is s-edge hamiltonian.*

Note that condition $\mathcal{L}_{n,s}$ is weaker then Ore condition.

Later Skupień and Wojda proved that the condition $\mathcal{L}_{n,s}$ is sufficient for a graph to have a stronger property (for details see [7]). Wojda [9] proved the following Ore type theorem:

Theorem 1.5 *Let G be a graph on* $n \geq 3$ *vertices, such that for every pair of nonadjacent vertices x and y*

$$\mathrm{d}(x,G)+\mathrm{d}(y,G)>\frac{4n-4}{3}.$$

Then every matching of G lies in a hamiltonian cycle.

In 1996 G. Gancarzewicz and A. P. Wojda [4] proved the following Fan type theorem:

Theorem 1.6. *Let G be a* 3*-connected graph of order* $n \geq 3$ *and let M be a k-matching in G. If G satisfies* $\mathrm{P}(n+k)$:

$$\mathrm{d}(x,y)=2 \;\Rightarrow\; \max\{\mathrm{d}(x),\mathrm{d}(y)\} \geq \frac{n+k}{2}$$

for each pair of vertices x and y in G, then M lies in a hamiltonian cycle of G or G has a minimal odd M-edge cut-set.

In this paper we shall find a Fan type condition under which every path system of length k in a graph G lies in a hamiltonian cycle.

For notation and terminology not defined above a good reference should be [2].

2. RESULT

Theorem 2.1. *Let G be a graph on* $n \geq 3$ *vertices and let S be a path system of length k with l internal vertices which components are paths of length at most 2 such that if* $P:\ u_1u_2u_3 \subset S$ *and* $\mathrm{d}(u_1,G), \mathrm{d}(u_2,G) \geq \frac{n+k}{2}$ *then* $\mathrm{d}(u_3,G) \geq \frac{n+k}{2}$ *. If G is* $(l+3)$*-connected and G satisfies* $\mathrm{P}(n+k)$:

$$\mathrm{d}_G(x,y)=2 \;\Rightarrow\; \max\{\mathrm{d}(x,G),\mathrm{d}(y,G)\} \geq \frac{n+k}{2}$$

for each pair of vertices x and y in G, then S lies in a hamiltonian cycle of G or the graph $\widetilde{G}$ *has a minimal odd* M_S*-edge cut-set.*

Note that under assumptions of Theorem 2.1 we have $0 \leq l \leq \lfloor \frac{k}{2} \rfloor$.

It is clear that Theorem 1.6 is a simple consequence of Theorem 2.1.

3. PROOF

Proof of Theorem 2.1.

Consider G and S as in the assumptions of Theorem 2.1.

We can now define the set A:

$$A = \{x \in V(G) : \mathrm{d}(x, G) \geq \frac{n+k}{2}\}.$$

Note that if x and y are nonadjacent vertices of A then the graph obtained from G by the addition of the edge xy also satisfies the assumptions of the theorem. Therefore and by Theorem 1.1 we may assume that:

$$xy \in E(G) \quad \text{for any} \quad x, y \in A \quad \text{and } x \neq y. \tag{3.1}$$

By (3.1) A induces a complete subgraph $G(A)$ of the graph G. Let $GV \setminus A$ be a graph obtained from G by deletion of vertices of the graph $G(A)$ (i.e. vertices from the set A).

Now consider a component D of the graph $GV \setminus A$.

Suppose that there exist two nonadjacent vertices in D. Since D is connected we have two vertices x and y in D such that $\mathrm{d}G(x,y) = 2$ and by the assumption that G satisfies $\mathrm{P}(n + k)$ we have $x \in A$ or $y \in A$, a contradiction.

So we can assume that every component of $GV \setminus A$ is a complete graph K_ι, $\iota \in I$, joined with $G(A)$ by at least $l + 3$ edges.

If $K_{\iota 0}, K_{\iota 1} \in \{K_\iota\}_{\iota \in I}$ are such that $\iota_0 \neq \iota_1$ then:

$$N(K_{\iota_0}) \cap N(K_{\iota_1}) = \emptyset. \tag{3.2}$$

In fact, suppose that $N(K_{\iota 0}) \cap N(K_{\iota 1}) \neq \emptyset$. Then we have a vertex $y \in K_{\iota 0}$ and a vertex $y' \in K_{\iota 1}$ such that $\mathrm{d}G(y,y') = 2$ and by $\mathrm{P}(n + k)$ either $y \in A$ or $y' \in A$. This contradicts the fact that $K_{\iota 0}$ and $K_{\iota 1}$ are two connected components of $GV \setminus A$.

If $C \subset G$ is a cycle in G then be $GV \setminus C$ we denote a graph obtained from G by deletion of vertices of the cycle C.

The graph G consists of a complete graph $GV \setminus A$ and of a family of complete components $\{K_\iota\}_{\iota \in I}$.

Let $K \in \{K_\iota\}_{\iota \in I}$.

Let $P : u_1 u_2 u_3$ be a path of length 2 from S. P is called a A-ear if u_1, $u_3 \in A$ and $u_2 \in \mathrm{V}(K)$, and respectively a K-ear if u_1, $u_3 \in \mathrm{V}(K)$ and $u_2 \notin \mathrm{V}(K)$ (in this case $u_2 \in A$).

If $\mathrm{E}(K,A) \cap \mathrm{E}(S) \neq \emptyset$, then in $\mathrm{E}(K,A) \cap \mathrm{E}(S)$ we can have a family of ears and a number of edges from $\mathrm{E}(S)$ which does not form any ear.

Now we shall define a cycle C. First consider a path containing only all A-ears. Next we add to this path all remaining vertices from A and all edges from the set $\mathrm{E}(S) \cap \mathrm{E}(G(A))$. All those edges and vertices form the cycle C.

Note that the cycle C performs the following conditions:

- C contains all edges of $\mathrm{E}(S) \cap \mathrm{E}(GV \setminus A)$ and all vertices of A. (3.3)
- If $K_{\iota 0}$ and $K_{\iota 1}$ are two different components of $GV \setminus C$ then
  $N(K_{\iota 0}) \cap N(K_{\iota 1}) = \emptyset$. (3.4)
- Let $x \notin V(C)$, $y \in V(C)$ and $xy \in E(G)$ then: (3.5)
  if y is not an internal vertex of S, then $y \in A$,
  if y^- is not an internal vertex of S, then $y^- \in A$,

if y^+ is not an internal vertex of S, then $y^+ \in A$.

Such cycle exists since $GV \setminus A$ is a complete graph and G satisfies (3.2).

Hence G is $(l + 3)$-connected, every component of $GV \setminus C$ is a complete graph joined with C by at least 3 edges which ends are not internal vertices of S.

Let K be a connected component of $GV \setminus C$.

We shall show that we can extend the cycle C over all vertices of K, over all edges of S in K and over all edges of S joining K with C preserving the properties (3.3) — (3.5) or that the graph $\widetilde{G}$ has a minimal odd M_S-edge cut-set.

Case 1

Among the edges joining K with C there are no edges from path system S. Since G is $(l + 3)$-connected K is joined with C by at least 3 edges which ends are not internal vertices of S. We have x_iy_i such that $x_i \in K, y_i \in C$ and $y_i^-, y_i, y_i^+ \in A$, for $i = 1,2,3$. We can assume that vertices x_1, x_2, are joined by one path P from S. Here P is directed from x_2 to x_1.

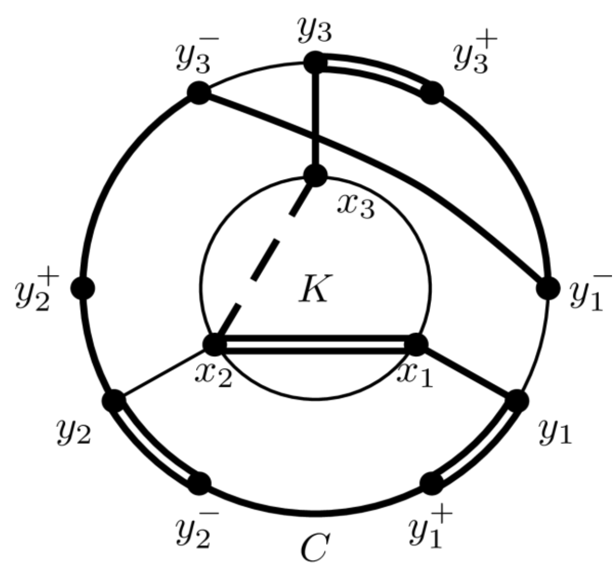

Figure (3.1).

Suppose that also $y_1y_1^+$, $y_2y_2^- \in$ E(S). If $y_3y_3^+ \in$ E(S) (then y_3 is a start vertex of one path from S directed towards x_1) we can consider the cycle (see Figure (3.1)):

$$C' : \quad y_3x_3v_1 \ldots v_kPy_1y_1^+ \ldots y_2^- y_2y_2^+ \ldots y_3^- y_1^- \ldots y_3^+, \tag{3.6}$$

where $v_1 \ldots v_k$ is a path containing all remaining vertices from the set

$$\mathrm{V}(K) \setminus (\mathrm{P} \cup \{x_3\})$$

and edges from the set (E(S) ∩ E(K)) \ E(P).

when $y_3^- y_3 \in$ E(S) we can carry out similar construction of cycle C'.

$$C' : \quad y_3 x_3 v_1 \ldots v_k P y_1 y_1^+ \ldots y_2^- y_2 y_1^- \ldots y_3^+ y_2^+ \ldots y_3^- .$$

Note that we can do the same if y_1 and y_2 joined by one path from S.

If y_2 and y_3 are end vertices of the same path from S or $y_2 y_2^+$, $y_3 y_3^- \in$ E(S) we can carry out a similar construction.

Suppose that $y_1 y_1^+ \in$ E(S). We can consider the cycle:

$$C' : \quad y_3 x_3 v_1 \ldots v_k P y_1 y_1^+ \ldots y_2^- y_2 y_1^- \ldots y_3^+ y_2 \ldots y_3^- .$$

Supposing that $y_1^- y_1$, $y_3^- y_3 \in$ E(S) a good extension of C should be the cycle:

$$C' : \quad y_3 x_3 v_1 \ldots v_k P y_1 y_1^- \ldots y_3^- y_1^+ \ldots y_2^- y_2 \ldots y_3^- .$$

The last two cycles are good also if y_2 and y_3 are end vertices of the same path from S. when $y_i y_1^+ \in$ E(S) for $i = 1, \ldots, 3$ we can define C' as in (3.6).

It is clear that the new cycle C' fulfills (3.3) — (3.5) and is an extension of C such that

$$\mathrm{V}(C) \subset \mathrm{V}(C') \quad \text{and} \quad ((\mathrm{E}(C) \cup \mathrm{E}(K)) \cap \mathrm{S}),\ (\mathrm{E}(C,K) \cap \mathrm{S}) \subset \mathrm{E}(C'). \qquad (3.7)$$

If among the edges joining K with C there are no edges from path system S then all other situations can be reduced to those presented above.

Case 2

Among the edges joining K with C there are some edges from path system S.

Since G is (l + 3)-connected K may be joined with C by a family of K-ears and at list three edges which ends are not internal vertices of S.

Hence G and S satisfies the following condition: if $P: u_1 u_2 u_3 \subset S$ and d(u_1,G), d(u_2,G) $\geq \frac{n+k}{2}$ then d(u_3,G) $\geq \frac{n+k}{2}$ edges from E(C,K) ∩ E(S) may be as on Figure (3.2).

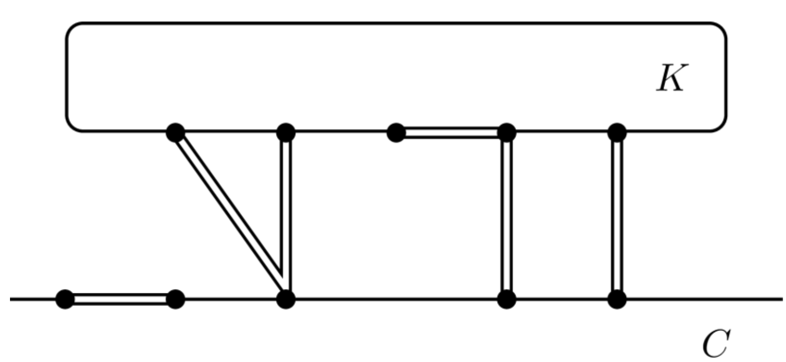

Figure (3.2).

The graph $\widetilde{G}$ has a minimal odd M_S-edge cut-set if there is a component K of $GV \setminus C$ which is joined with cycle C only by an odd number of edges from E(S) or an odd number of edges from E(S) and edges from E(G) \ E(S) with at least one end vertex in the set of internal vertices of path system S. In those cases the theorem is proved, so we may assume that $\widetilde{G}$ has no minimal odd M_S-edge cut-set

Subcase 2.1.

Among edges joining K with C we have an even number of edges from E(S), say $s = 2r$, ($r \geq 1$) which does not form any ear.

So we have vertices $x_1,\ldots,x_{2r} \in K$ and $y_1,\ldots,y_{2r} \in C$ such that $x_iy_i \in$ E(S), for $i = 1,\ldots,2r$. We can assume that each edge x_iy_i is in path of length 2 from path system S. Then we have vertices $x_i^+ \in$ V(K) such that $x_i^+x_i \in$ E(S), for $i = 1,\ldots,2r$.

Let u, $v \in$ V(C) be such that all edges from E(C,K) ∩ E(S) lying between u and v belong to some ears. In the cycle C we have a path W : $uc_1\ldots c_kv \subset C$. We shall define a new path $Q(u,v)$. If u and v are not in any ear. The path $Q(u,v)$ is a path joining u with v such that E(W) ∩ E(S) ⊂ E($Q(u,v)$) and $Q(u,v)$ contains all c_i such that c_i is not an internal vertex of a K-ear. In other words $Q(u,v)$ arises from W by removing internal vertices of all K-ears. It is possible because if c_i is an internal vertex of a K-ear then c_i^-, $c_i^+ \in A$.

When u is internal vertex of a K-ear, then we start the path $Q(u,v)$ from the first vertex c_i which is not internal vertex of any K-ear. If v is internal vertex of a K-ear, then we end the path $Q(u,v)$ from the last vertex c_i which is not internal vertex of any K-ear.

The construction of $Q(u,v)$ is shown on figures (3.3) — (3.5).

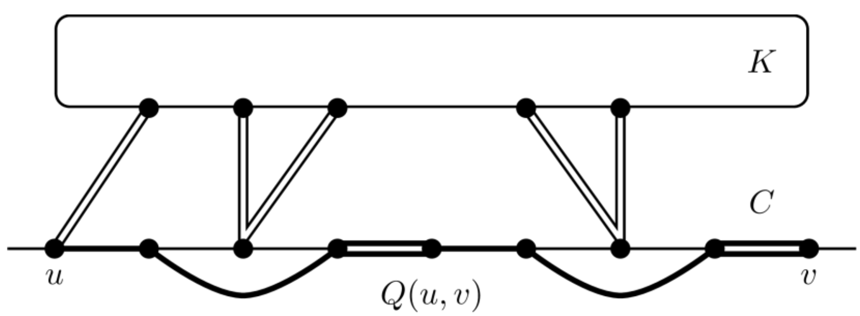

Figure (3.3).

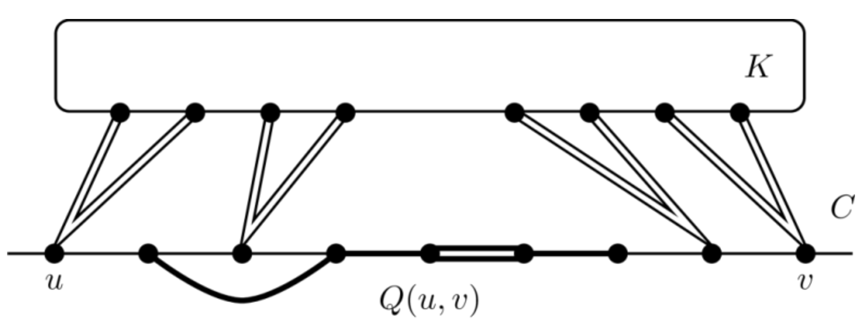

Figure (3.4).

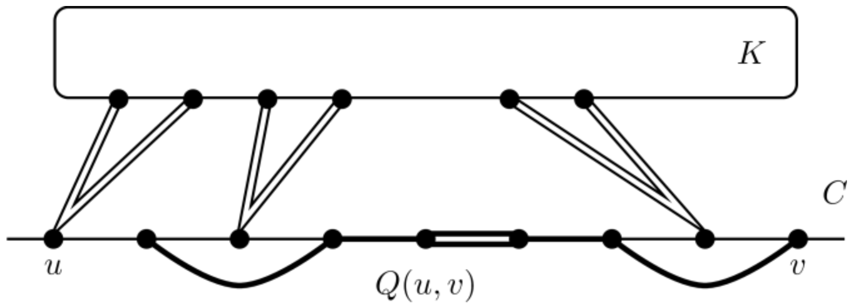

Figure (3.5).

First consider the path P_K containing only all K-ears. Now we can define the extension of the cycle C as follows (see Figure (3.6) (for $r = 2$)):

$$C' : \quad \begin{array}{l} y_1 x_1 x_1^+ P_K x_2^+ x_2 Q(y_2, y_1^+) Q(y_2^+, y_3) x_3 x_3^+ \ldots y_{2r-1} \\ x_{2r-1} x_{2r-1}^+ v_1 \ldots v_s x_{2r}^+ x_{2r} Q(y_{2r}, y_{2r-1}^+) Q(y_{2r}^+, y_1), \end{array}$$

where $x_{2r-1}^+ v_1 \ldots v_s x_{2r}^+$ is a path containing all remaining vertices of K and edges of $E(S) \cap E(K)$, this path exists because K is complete.

It is clear that the new cycle C' fulfils (3.3) — (3.5) and (3.7).

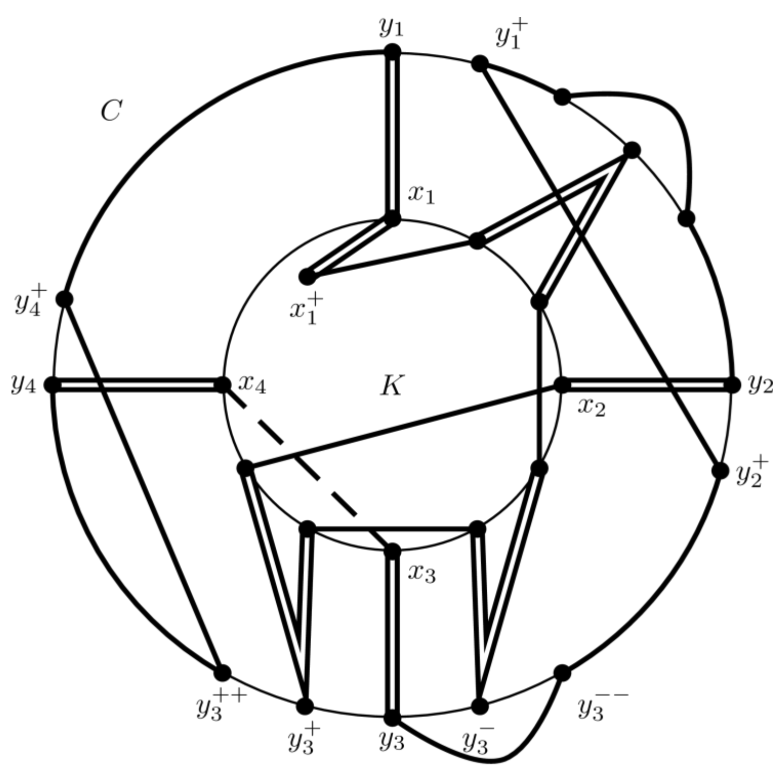

Figure (3.6).

Subcase 2.2.

Among edges joining K with C we have an odd number of edges from E(S), say $s = 2r - 1$, ($r \geq 1$) which does not form any ear.

So we have vertices $x_1,\dots,x_{2r-1} \in$ V(K) and $y_1,\dots,y_{2r-1} \in$ V(C) such that $x_iy_i \in$ E(S). We can assume that each edge x_iy_i is in path of length 2 from path system S. Then we have vertices $x_i^+ \in$ V(K) such that $x_i^+x_i \in$ E(S), for $i = 1,\dots,2r - 1$.

Since we have assumed that $\tilde{G}$ has no minimal odd M_S-edge cut-set we have at least one edge say xy, ($x \in K$, $y \in C$) such that $xy \notin$ E(S), x and y are not an internal vertices of S.

We shall consider four subcases according as x or y are extremities of an edge from the set E(S).

Suppose that $y \notin \{y_1,\dots,y_{2r-1}\}$ and $x \notin \{x_1,\dots,x_{2r-1}\}$. In this case we have a vertex $y_{i0} \in V$ (C), ($i_0 \in \{1,\dots,2r - 1\}$) such that on the cycle C the vertices are ordered as follows: $y_{i0}\dots y\dots y_{i0+1}$.

Consider a path $xv_1\dots v_sx_{i0+1}{}^+x_{i0+1}$ containing all vertices from the set

V(K) $\setminus\{x_1,x_1{}^+,\dots,x_{i0},x_{i0}{}^+,x_{i0+2},x_{i0+2}{}^+,\dots,x_{2r-1},x_{2r-1}{}^+\}$ all K-ears and all edges from E(S) $\cap$ E(K).

If $y^-y \in$ E(S) consider the following cycle C' :

$$C': \quad \begin{array}{l} y^-yxv_1\dots v_sx^+_{i_0+1}x_{i_0+1}Q(y_{i_0+1},y^+)Q(y^+_{i_0+1},y_{i_0+2})x_{i_0+2}x^+_{i_0+2}x^+_{i_0+3}x \\ \qquad Q(y_{i_0+3},y^+_{i_0+2})Q(y^+_{i_0+3},y_{i_0+4})\dots Q(y_{i_0},y^+_{i_0-1})Q(y^+_{i_0},y^-), \end{array}$$

satisfying properties: (3.2) — (3.5) and (3.7).

when $r = 1$ the edge xy must be independent with all x_iy_i, so now we have $r \geq 2$.

Suppose that for $y \notin \{y_1,\dots,y_{2r-1}\}$ and there is an $i_0 \in \{1,\dots,2r - 1\}$ such that $x = x_{i0}$. In this case $x_{i0}x_{i0}{}^+ \notin$ E(S).

If $yy^- \in$ E(S) then we define a new cycle $\tilde{C}$ as follows:

$$\tilde{C}: \quad y^-yx_{i_0}Q(y_{i_0},y^+)Q(y^+_{i_0},y^-).$$

and consider the complete graph D obtained from K by deletion of the vertex x_{i0}.

D is a component of $G_{V\setminus\tilde{C}}$. Note that $\tilde{C}$ and D satisfies conditions (3.3) — (3.5) and (3.7). Since $r \geq 2$ D is joined with $\tilde{C}$ by an even number of edges from E(S), which does not form any ear and then we can proceed as in subcase (2.1).

Suppose that for some $i_0,j_0 \in \{1,\dots,2r - 1\}$ $x = x_{i0}$, $y = y_{j0}$, and ($i_0 \neq j_0$).

First consider the case $r = 2$ and vertices y_1, y_2, y_3 are ordered in C as follows: $y_1\dots y_2\dots y_3$.

We can assume that $y = y_1$, $x = x_3$ ($x_3x_3^+ \notin$ E(S)) and then consider the cycle:

$$C': \quad y_3x_3y_1x_1v_1\dots v_sx_2Q(y_2,y_1^+)Q(y_1^-,y_3^+)Q(y_2^+,y_3),$$

where $x_1v_1\dots v_sx_2$ is a path containing all remaining vertices from K all K-ears and all edges from E(S) $\cap$ E(K).

Again the cycle C' has properties: (3.2) — (3.5) and (3.7).

when $r > 2$ we have $y_l x_l \in E(S)$ and we assume that in the cycle C vertices are ordered as follows: $y_{j0}...y_l...y_{i0}$. Now we can define a new cycle $\tilde{C}$:

$$C' : \quad y_{i_0}x_{i_0}y_{j_0}x_{j_0}x_l^+x_lQ(y_l,y_{j_0}^+)Q(y_{j_0}^-,y_{i_0}^+)Q(y_l^+,y_{i_0}).$$

and consider the complete graph D obtained from K by deletion of the vertices x_{i0}, x_l and x_{j0}.

D is a component of $G_{V\setminus\tilde{C}}$. Note that $\tilde{C}$ and D satisfies conditions (3.3) — (3.5) and (3.7). Since $r > 2$ D is joined with $\tilde{C}$ by an even number of edges from E(S), which does not form any ear and a family of ears, so we can proceed as in subcase (2.1).

Subcase 2.3.

Among edges from E(S) joining K with C we have only edges which are forming K-ears.

Hence G is $l + 3$ connected we have also at least 3 edges from E(G) \ E(S) which ends are not internal vertices of S.

This case is similar to the case 1. The only difference is fact that we have K-ears, but using paths $Q(u,v)$ we can extend the cycle as in case 1.

In all cases we have extended the cycle C, so the proof is complete.

4. CONCLUSIONS

The proof of Theorem 2.1 is an example of application of the closure technique. Note that the construction of the cycle C in the closure of the graph G is algorithmic but unfortunately it is possible that the cycle is using edges that does not belong to the initial graph.

Our result is an extension of Theorem 1.6.

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