Available online at www.worldscientificnews.com

World Scientific News

An International Scientific Journal

WSN 127(1) (2019) 1-55 EISSN 2392-2192

Wave-particle duality of a 6D wavicle in two paired 4D dual reciprocal quasispaces of a heterogeneous 8D quasispatial structure

Jakub Czajko

Science/Mathematics Education Department,
Southern University and A & M College, Baton Rouge, LA 70813, USA

E-mail address: sunswing77@netscape.net

ABSTRACT

According to Louis de Broglie, wave-particle duality can imply presence of a single entity also known as wavicle. Hence, instead of the double solution he proposed to explain the duality, I depict the 6D wavicle in a multispatial mathematical entity represented in two 4D quasigeometric heterogeneous structures comprising two paired 3D dual reciprocal spaces, one of which represents the particle and the other the wave that was supposed to guide the particle in his pilot wave theory. The operational nature of the (2×4)D = 8D biquaternionic quasigeometric structure of the wavicle is synthesized from twin operations performed over paired 3D homogeneous dual reciprocal spaces, each of which is immersed in a pair of two 4D asymmetrically overlapping heterogeneous quasigeometric (3+1)D = 4D spatial structures.

***Keywords*:** Wave-particle duality, wavicle, multispatial mathematical reality of physics

1. INTRODUCTION

Mathematicians tend to ascribe great importance to the elegance of their methods and to the beauty of their results. This cosmetic strive is justified by efficiency of mathematics, for it apparently benefits the economy of thinking [1] among other reasons of lesser importance.

(Received 12 March 2019; Accepted 10 April 2019; Date of Publication 11 April 2019)

Their emphasis is on abstract thinking about physical reality instead of on truthful depiction of the reality. The truthfulness is implicitly presumed as naturally emerging from their artlike reasonings. But history of development of mathematical ides, theories and methods is littered by conceptual failures due to misapprehension of the Nature caused by almost statutory and arrogant disregard for feedback provided by curious results of formerly quite unanticipated experiments. The disregard affected not only the perfunctory pure mathematics but created obstacles also to prospective development of physical theories in search for scientific truths.

For Poincaré, geometric space is just a form of our understanding [2] p. 11f, and thus the form could already exist before the geometric substance could fill the form, whereas for Helmholtz and Lie, for instance, the structural substance should exist before the form [2] p. 60. For Poincaré, therefore, scientific laws of Nature are not imposed on us by the Nature herself but are imposed by us on [our purported understanding of] the Nature [2] p. 20. Hence pure mathematics could have created an elegant abstract mathematical reality, which may not be able to actually exist in the real physical world we live in. That is what happened with some abstract mathematical concepts that ignore feedback from physical experiments, indeed.

Perhaps we should devise abstract mathematical notions based upon suggestions hinted at by formerly unanticipated yet unbiased curious experimental results, if we want to keep mathematics in sync with how the Nature seems to actually work. For inability to predict (or just to anticipate) curious experimental results can be seen as indicative of inappropriateness of the former mathematics that somewhat diverged from the ways the Nature apparently operates. Moreover, if our mathematical operations do not fit the structures hinted at by what the Nature reveals in the unanticipated experimental results, then perhaps her structural features may also have been somewhat misunderstood. If mathematics should predict natural phenomena found in the physical reality we live in [3], [4] then algebraic operations should fit the geometric structures we want to operate on, and vice versa.

Both these views could be complementary, i.e. right, even though not always complete or not quite correct. Hence the thoughts of Poincaré and of Helmholtz-Lie could both be helpful indeed, though perhaps on quite different stages of development of mathematical or physical theories. Since we created our theoretical notions by observing natural phenomena [5], [6] and the forms in terms of which we can much better describe the observed natural phenomena and then make predictions, which then could be tested – whether physically or perhaps just theoretically – and after that, by refining both the terms and the forms we may develop even better, or just more comprehensive, theories. Then the envisioned process of incremental improvement could be subsequently repeated on next higher conceptual levels. For the [structural] pure mathematics results from physical geometries by deinterpretation [7]. Recall that before variables can be interpreted as [ones representing] 'objects', the domain of the interpretation is to be the domain of objects over which the variables are going to range [8]. I should not axiomatize any properties before I am able to unambiguously classify and identify major objects under investigation. This approach runs contrary to the usual prescription for fixing/defining them right up front via axioms, with all features of the objects involved.

Axiomatic approach worked fine with trivial objects. But for conceptually complicated objects, such as energy, potentials and their respective host-spaces – not just abstract sets – the Euclid-like axiomatization may not always work, for it is prone to quite inadvertent "creation" of nonexistent objects, or perhaps even "proving" existence of some impossible to exist, hence illusionary, abstract objects. Some authors go so far as to claim that the mathematical reality of existential problems is either profoundly real or absurdly surreal [9].

No matter how abstract it is designed to be, the reasonable conceptual mathematics should always be in sync and comply with the actual, experienced workings of the Nature.

In 1939 MacLane [in his own words] has attempted to characterize algebra as the study of structure of systems defined by suitable postulates placed on rational operations, but in 1963 that attempt appeared to him "naive and somewhat limited" [10]. Thus even some categorizing algebraists honestly admitted that postulating existence of abstract algebraic structures via suitably concocted axioms is an exercise in futility, once they grew up mathematically. I am not against theory of categories but am aware of some of its – too simplistic to be true in general, i.e. regardless of the espoused underlying structural paradigms – conclusions. Since the fundamental object of mathematics is not a set composed of some elements, but a sheaf of functions on some nonspecified space or locale [11], then perhaps we should first synthesize properties of the space itself from the sheafs' behavior instead of investigating the predetermined geometrical objects placed in an already predefined space endowed with already fixed properties, which approach cannot guarantee any completeness of such an investigation. Furthermore, perhaps it would be desirable to establish geometry from mirror reflection without specifying a certain dimension [12].

Imprecise assertions can confuse even mathematicians who, sometimes unwittingly, start investigating the contents of their own perplexed minds [13], [14], [15] p. 144, for qualitative mathematical relationships are not only the results of an abstract power of the mind, but also of human activities [16]. Thinking about an object can alter the object's perceived attributes or even add a few nonexistent attributes during the process of idealization [17], when – on a higher level of abstraction – one begins to invent fictional linguistic entities [18]. Therefore, we must avoid making any existential postulates – whether these are made explicitly or are disguised as allegedly innocent "definitions" – for they are absolutely inadmissible (as tacitly supplying often imperceptibly destructive additions) in constructive mathematical reasonings.

Only through matching of operations to the – corresponding to them – geometric or quasigeometric structures one can ensure the viability of existence of both, which then could be confirmed in some experiments conducted preferably outside of the purely mathematical framework in which they were created. For some internal inconsistencies of mathematics can be buried so deep that it could take a few centuries to uncover them [19].

However, because mathematical objects exist as essentially normative systems [20], mathematics – conceived as an abstract formalized calculus – could indeed be developed freely [i.e. quite independently of the real world], according to its own internal rules/laws [21]. Nevertheless, even if underdefined on purpose, operational/algebraic procedures as well as geometric and quasigeometric structures/objects need a definite frame of reference such as an algebraic or geometric basis or both, in which these objects could be unambiguously represented. Although there is no *a priori* restriction on the character of any algebraic basis, it is clear that only homogeneous basis could ensure unambiguity of representations, even though most physically meaningful situations involve various objects of different kinds and thus require heterogeneous bases. Hence without denying the need for deployment of heterogeneous bases in the world we live in, synthetic approach to mathematics prefers the use of homogeneous subbases, which could then be combined into a heterogeneous basis. The latter, in turn, demands multispatial approach to geometric and/or quasigeometric structures. Thus, synthetic mathematics compels us to use multispatial approach in physics too.

The multispatial synthetic approach to mathematical and physical theories relies on quite distinct and conceptually different set of paradigms underlying the prospective new synthetic mathematics. It is vitally important to realize that interpretation of the same set of axioms and primitive notions could be affected by the – oftentimes unmentioned – paradigms. Traditional mathematics routinely disregarded the influence of paradigms on interpretations, however.

Now the other reason for distinguishing homogeneous from heterogeneous bases is that under the usually unspoken single-space reality (SSR) paradigm, which tacitly assumes that everything happens in a kind of mathematical universe that resembles a single set-point space, there was no absolute necessity to assign bases to spaces as well as to representations. In the new synthetic approach to mathematics that emphasizes the need to see spaces as entirely distinct views of the same underlying reality, each of these views (and thus also the spaces) require its own native basis, whether it is an algebraic or a geometric or a quasigeometric one. Bases of vector spaces were discussed in both set-theoretical and an algebraic setting in [22]. Homogeneous coordinates were discussed in classical geometric setting in [23]. Abstract representations in homogeneous spaces were discussed in [24].

My new synthetic approach to mathematics and physics was influenced not as much by philosophy as by René Thom who realized that at the foundations of all major scientific disciplines one finds some phantasmagories or fundamental oppositions [25], which may appear as illusions or just apparitions to some, but they can also have a certain conceptual or at least inspirational value/merit. Volatility of their merits arises not because our thinking is always inherently faulty, but rather because it is always framed in (and often constrained by) our ever-changing paradigms even if their continuous change is imperceptible. My critique is focused on synthesizing various merits rather than on enthroning one view/merit/value at the expense of other views. None of the presently cherished views is final. It is only a matter of time for a better idea to emerge and change the game by subduing the now prized ideas. Truth is not electable. We cannot afford to discard the inconvenient ideas we do not yet fully grasp.

The synthetic approach to mathematics entails matching of procedures, i.e. differential and algebraic operations, to constructions of the corresponding to them geometric or quasigeometric structures, the latter resembling operational procedures.

Algebraic constructions and their extensions were discussed in [26] and differential operators on algebraic varieties in [27]. This new synthetic approach demands acceptance of a multispatial reality (MSR) paradigm, which should replace the old SSR paradigm. However, implementation of the MSR paradigm requires more finegrained defined operators, both algebraic/nondifferential and differential, for these kinds of operators are not always exclusive. Even the differential operator nabla is not a self-contained entity, for its effect depends on its usage [28]. Comprehensive critique of the nabla operator was given in [29] – compare also the so-called "second fundamental confusion of calculus" [30] p. 189. This is the chief reason for redefining the operators in more detail, which shall be done elsewhere.

2. HETEROGENEOUS BASES

While vectorial evaluations can simplify calculations involving differentials, Tait already remarked that quaternions have slight advantage over vectors, for unlike vector product of vectors, quaternions retain associativity, for $j \times (k \times k) = 0 \neq (j \times k) \times k = - j$ in the (j,k,l) basis frame [31]. The usual imaginary unit i shall be used as an algebraic operator henceforth.

However, most physical situations involve quantities of distinct nature which may require deployment of definitely heterogenous bases. This is often unavoidable not only for the sake of unambiguous structural representations but also for performing successful operations on independently varying physical magnitudes, which may not always be treated as parameters housed in a single homogeneous space. Yet because operating on objects depicted in a heterogenous basis can yield questionable results, I prefer – wherever it is possible – to split the heterogenous operating procedures into homogeneous suboperations performed over multiple spaces or quasispatial structures, each of which must be equipped with homogeneous basis, whether it is algebraic or geometric one. The main reason for doing this splitting of operations is to preserve their orthogonality which can be ensured within homogeneous bases. Since numbers used to be treated as entities interchangeable with their values – even though their value is just one of several attributes of the entities called numbers – there was no conceptually consistent approach to spaces viewed as geometric or quasigeometric structures as opposed to mere sets that are treated as arbitrary agglomerations of number-points.

Hausdorff initiated foundations of cleavability of sets and defined also the (set-theoretical) notion of coherence based upon mappings of point-sets, which although useful [32], could cast a shadow of distrust on the idea of separable spaces (which he regrettably called "incoherent") due not only to the unfortunate connotation of the latter qualifier but also because of their misperceptions present in set theory itself.

Mappings are acceptable under the SSR paradigm but treating multispatial structures in terms of mappings could be ambiguous. Since mappings map structurally defined domain into/onto (often operationally defined) range, interspatial mappings could destroy the homogeneity of representations. I am not against mappings but am aware of oversimplifications due to tacit existential postulates the mappings could introduce. I am also for correctness of operations and clarity of concepts.

Another reason for splitting heterogenous basis into chained homogeneous bases is that polynomials of degree higher than 4^{th} are in general unsolvable by radicals (per Galois and Abel) [33]. Some simple practical aspects of algebraic solutions by radicals were briefly explained in [34]. Methods using resolvents, some of which can appear as supplying viable solutions [35], can make the solutions even more difficult rather than simpler for higher than 4^{th} degrees [33], gave me yet another reason for seeking the paradigm shift from SSR to MSR because reasonable geometric dimensionality is not only capped at 4D but also permits presence of overlapping quasigeometric multispatial structures [33], [36]. By "reasonable" approach to dimensionality I mean such that matches operational procedures to constructible geometric or quasigeometric structures, for postulating existence of structures that cannot be constructed or operational procedures that cannot be solved makes no sense.

The other issue that is related to maintaining homogeneity of representations (including that in differentials) and thus constrains the formal representations of polynomials, is the number of variables known as Hesse's theorem, which was briefly mentioned and discussed in [37]. Various topological aspects of the unsolvability and nonrepresentability were discussed in [38].

Some topics of algebraic and geometric bases are discussed in terms of Clifford/geometric algebra in [39] p. 39, and in algebraic/hypercomplex setting in [40] pp. 20,24. Clifford algebras were discussed in abstract terms in [41] and also compared to Grassmann algebras in [42]. Orthonormal bases were well compared with nonorthonormal bases in [43].

Splitting of heterogeneous operations into homogeneous operations require multispatial approach to geometrical and quasigeometrical structures. However, the implementation of multispatial approach is contingent upon feasibility of performing unrestricted algebraic operations, including division by zero, which can be implemented in dual reciprocal spaces [44], [45]. Uninhibited division by zero is necessary also for operations performed in 4D spacetime [46].

The notions of duality and reciprocity (viewed as multiplicative inverse) are operationally related. Generalized geometric duality for hyperspaces S^n interchange points and hyperplanes [47-56]. Global duality was discussed in abstract terms in [57]. Note that S-duality not only interchanges the electric and magnetic roles but implies reciprocity too [58].

However, the MSR paradigm may require also some nonlinear relations/equations. Diagram of linear and nonlinear differential equations is shown in [59]. Reciprocal spaces as inverses of “regular” (or “real”) spaces, are often used in solid state physics, especially in crystallography, mainly because the value of wave vector $|\mathbf{k}| = k = 1/\lambda$ is expressed in inverse units of length for it is reciprocal to intervals of length such as the wavelength λ [60].

Change in regular space triggers an inverse change in the reciprocal space associated with the regular primary space for reciprocal spaces supply different representation/view of the same physical object. Shrinking (or enlarging) objects in the primary space appears as enlarging (or shrinking) representations of the objects in the reciprocal space. Reciprocity changes not the objects themselves but only their views/representations, providing thus the key for pairing of the regular/primary space P with its respective dual reciprocal space Q in geometric and/or abstract quasigeometric spatial structures. It is in the latter nontraditional sense that the reciprocity will be utilized in this paper. Multispatiality and reciprocity are tied together via paired dual reciprocal representations, which in conjunction with unrestricted division by zero permits evaluations such as $PP^* = 0 \cdot \infty = 1$ [44-46]. The asterisk denotes conjugation.

The term ‘inverse’ is used in two senses: as a reciprocal and as composite inverse [61] p. 76. Equating complex or hypercomplex inverse in the notation $f^{-1}(z)$ with reciprocal is clear abuse of language in tacit defiance of logic, for only the numbers’ modulus is truly multiplicatively inverted, but the angle (argument) is merely additively reversed, not inverted [61] pp. 5,11, [62]. Under the MSR paradigm we shall use the term ‘inverse’ as a multiplicative inverse. Although infinity is reciprocal to zero [61] p. 233, these two entities are operationally incompatible objects.

They cannot both fit into the same operational space even if they reside in the same set. Since complex and hypercomplex inverses involve both multiplicative and additive group operations, they demand the new synthetic approach to mathematics. I am not criticizing any particular book, for this evasively avoided nonsense is common to virtually all presentations of complex analysis, but one may still wonder why such – apparently inconvenient – truths are sidestepped instead of being at least addressed if not explained. It is presumably because these truths might reveal an elusive abstract reality concealed behind the officially recognized mathematical (as well as physical) reality.

The new synthetic approach to mathematics responds thus to three crises encountered in mathematics, which the past trends of logicism, formalism, and intuitionism were unable to overcome. Recall that these approaches to mathematics correspond to realism, nominalism, and conceptualism, respectively [63]. Since logicism and formalism can be discarded as lacking deep mathematical sophistication and often falling into the category of wishful thinking, only the failure of intuitionism seems worthy of careful review, which shall be done elsewhere. For the purpose of the present paper, we shall use mathematical intuition because it allows fast transition toward new forms of knowledge [64]. The new synthetic approach relies on intuitionism and constructivism, though without constrains of their contingencies. Truth in mathematics should be discovered via syntheses guided by experimental and/or theoretical hints, rather than be postulated through convenient yet often arbitrary axioms.

3. REVIEW OF SOME PRACTICAL ISSUES OF WAVE MECHANICS

It is known that direct measurements of the transition frequency ν between two quantum states whose energy difference is $\Delta E = h\nu$ are only feasible in the radio wave frequency, microwave and submilimeter wave regions of the whole frequency spectrum. But practical determination of the energy difference in the infrared, the visible and higher frequency regions is based on measurements of the wavelength λ according to the formula $\Delta E = hc/\lambda = hck$ [65] p. 1532. Since both, the Planck constant h and the speed of light in vacuum c are constant, it follows that the temporal frequency ν and the wavenumber $k = 1/\lambda$, i.e. "spatial frequency" reciprocal of the wavelength, can determine the energy difference. For energy and momentum are reciprocal [30] p. 503. In other words: Energy difference is always proportional to frequency. Whether it is the usual temporal frequency or spatial frequency is just a matter of practical determination. If so then the paradigm shift from SSR to MSR is not really an option but a must, for each space should be equipped with its own uniquely denominated orthogonal native homogeneous basis, whether it is an algebraic or a geometric one.

This theoretical hint raises thus – the conceptually curious though routinely overlooked – question: If the wavenumber representing both: the spatial frequency as well as energy difference has its own 3D length-based space (LBS), in which it can vary, then where is the – corresponding to it temporal – 3D time-based space (TBS), in which also the temporal frequency (as magnitude reciprocal to a certain elapsing time instant, or time interval τ such that $\nu = 1/\tau$) can vary? There is nothing wrong with the physics or mathematics but the latter appears somewhat incomplete until it would show where is the "temporal" geometric structure (i.e. temporal space) in which inverted meter (as the measure of wavenumber) turns into inverted second (cycle), a measure of temporal frequency? Since in physics these magnitudes are interchangeable, why was former mathematics averse to the idea of multispatiality? Maybe multispatiality might reveal something embarrassing about the mathematics.

Since elapsing time is not just an arbitrarily chosen running parameter – as both classical physics and quantum mechanics coach us to believe – but actually an independently varying magnitude too, then it should have its own native homespace, in which it could vary quite independently of all the other spatial magnitudes involved. For we always float with elapsing time no matter in which length-based spatial direction we can choose to go [66]. Clearly both the elapsing time and the temporal frequency, as well as the wavelength that is reciprocal of the spatial frequency, need their own homespaces each equipped with appropriately denominated native basis, in which these magnitudes could vary quite independently of all the other variables and parameters. Reciprocal sets of vectors are discussed in [67], [68].

This requirement for double treatment (temporal and spatial) of frequency with functional (i.e. nondifferential/integral) changes made to energy is the physical exhibit of our proceedings here. Function-based differential operators shall be discussed elsewhere. Synthetically speaking, this demand is a must when it is considered from an abstract, purely mathematical point of view, if abstract operational procedures should indeed fit/match the corresponding to them geometric or quasigeometric structures over which the procedural operations are supposed to be performed.

4. THE ESSENCE OF NEW SYNTHETIC APPROACH TO MATHEMATICS

Traditional development of pure mathematics was focused on abstract representations of mathematical objects and their relationships depicted within certain predefined sets or spaces and described in an artificially abstract language oftentimes designed to support unifying umbrella imposed on arbitrary operational procedures or postulated geometric structures.

This traditional approach worked fine for devising proofs of ever more sophisticated abstract classification schemas envisaged for the objects and for anticipated relationships existing between them. But unanticipated results of some unbiased physical experiments revealed the need for reevaluation of the heretofore successful classification schemas. The pinnacles of former ideas not only failed but sometimes even became obstacles on the road to expound the curious experimental results that apparently were blowing up some of the previous methods and ideas. Notice that in the essentially algebraic setting of category theory the term 'structure' is implicit and is routinely treated as a primitive concept [69]. While modifying arguments to account for changed circumstances in new proofs seems easy, transferring methods to new or significantly different situations can be problematic [70].

Instead of trying to find a better representation of an object defined within the allegedly "proven" though conceptually failing old classification schemas, I shall opt for discovering a new underlying mathematical reality based upon the mathematically inconvenient objects while determining their features on the fly. Since there are only two basic kinds of analyses: 1) the problem to prove, and 2) the problem to find something unknown [71], the synthetic approach is essentially abstract mathematical analysis of the second kind. The new synthetic approach emphasizes discovery rather than investigating features of predefined notions, for definitions are often mere conveniently phrased existential postulates in disguise [72], [73].

Although many authors agree with the Poincaré's conviction that all geometrical theories are conventional and that their axioms are merely definitions in disguise, some authors are rather uncomfortable with the fact that the – assigned by often arbitrary convention – axioms could tacitly (mis-)define the (sought for) abstract reality (if it is to be operationally effective) of the arbitrarily chosen geometry, sometimes pushed in defiance of the actual physical reality we live in [74]. Perhaps there is yet another reality behind the physical reality and if so then the heretofore ignored extra reality is mathematically just as necessary as the physical reality is. Notice that I am not trying to postulate the extra reality into existence but am saying that our mathematics virtually seems to support two interconnected kinds of reality at once, the actual presence of one of which was routinely refused to be acknowledged.

It was already experimentally confirmemd that facts established by various observers cannot coexist within a single observer-independent framework [75]; see also [76], [77]. All this subtly implies the necessity of deploying at least virtual, if not actual, multispatiality.

Although some authors consider violations of Bell's inequalities as a proof that quantum mechanics is complete – see overview in [78], a complete representation in not necessarily quite complete setting does not really mean full completeness but only relative completeness, which would amount to incompleteness once the present setting is expanded into a new theoretical territory. The current experimental violations of Bell's inequalities cannot be the final proof [of completeness, nor even of validity] of present quantum physics [79], indeed. For comprehensive exposition of Bell's theorem and some related issues see [80].

Although we can deduce internal properties of abstract objects, mathematics cannot make deductions regarding existential conclusions determining the prospective appearance of the mathematical reality itself. Reliance on deductive proofs is only of purely internal validity. Since proofs can be invalidated by just altering some paradigms, deductions venturing into external nature or characteristics of mathematics can serve only as theoretical hints, requiring thus an external experimental confirmation. No science can determine itself. This is true for mathematical and all other theories. Disregard for this “No selfdetermination” principle is the main reason why some mathematical theories [81] (and mathematical underpinnings of some physical theories [82-84]) became corrupted. Therefore verification of validity of mathematical concepts and theories should come from unbiased physical experiments.

Neither could physics itself make its own deductions regarding the existential conclusions that supposedly define the philosophically preferred appearance of prospective nature of the physical reality. Case in point: Extending electromagnetic theory from 2D to 3D – hence from rotations in plane governed by the group O(2) to 3D rotations governed by the group O(3) – produces quantization of charge from geometry, resulting in nonabelian electromagnetics [85] p.68ff. In the latter case the nominally massless photon should have mass nonetheless [86] pp. 5,117ff, [85] pp. 44,107,123,130ff, [87], which is quite legitimate a conclusion. Postulating massless photon because of its wavelike behavior is defeated by altering an abstract paradigm. Electron has mass and also wavelike nature, which was experimentally confimed [88], [89].

Therefore no theory of physics should ever try to selfdetermine itself either. I have no intent to resuscitate de Broglie’s wave mechanics but to push some of his ideas through the obstacles that apparently stopped him. I shall not deal with former physics, neither with his double solution nor the Bohm’s continuation of turning the de Broglie’s pilot wave theory into a deterministic alternative to quantum theory, at least not at this time.

Since it was the set of faulty concepts of the previously underdeveloped mathematics that doomed de Broglie’s visionary ideas in physics I shall concentrate mainly on the faulty former mathematics. His physics will be used in the present paper as a guide to the prospective new mathematics. However, when it comes to proper conceptual understanding of the wave-particle duality this paper is closer to Bohr’s interpretation of the duality, who considered wave and particle as complementary concepts, which view de Broglie adopted despite explicitly considering it inadequate [90]. I will not cling to the de Broglie’s physical conceptualization of his idea that he eventually downsized into his pilot-wave theory.

The essence of the new synthetic approach is the recognition that no conceptually valid idea should be discarded right up front. Competing ideas should be synthesized with their contradictory (or complementary and even supplementary), back then, ideas in order to gain an unambiguous mathematical (though not really Hegelian) synthesis of those previously contending ideas, provided all the ideas to be synthesized were at least partly valid.

5. FROM QUANTUM-RELATIVISTIC ENERGY CLASH TO WAVE MECHANICS

Wave mechanics sprang from de Broglie’s attempt to reconcile relativistic and quantum-mechanical energy formulas, which appeared incompatible and not relativistically invariant:

$$m_0c^2 \neq h\nu_0 \;\;\Longrightarrow\;\; mc^2 = h\nu \tag{1}$$

and seemed inconsistent on the left-hand side (LHS) since the moving particle’s rest mass m_0 does not change in motion whereas the inner frequency ν_0 is getting smaller due to slowing down of the moving clocks. Here c is the speed of light in vacuum and h is the regular Planck's constant. De Broglie noticed, however, that the formula could become invariant in Galilean reference frames when written as $mc^2 = h\nu$ if it would refer to a stationary wave of frequency ν_0 treated as inner vibration of the wave, for now the frequency ν of this wave varies just as the mass m when the reference frame changes – compare [91] p. 18. Moreover, the LHS asserts a fundamental inertia of energy [92], a fact that should not be ignored.

Pursuing the aforesaid idea, de Broglie reportedly showed that if the phase wave $\phi(x,y,z,t)\exp[2i\pi\nu(t-(z/v_g))]$ satisfies d'Alembert equation then its amplitude ϕ obeys an equation, which resembles in form, though not exactly in its gist, Klein-Gordon equation

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -\frac{4\pi^2\nu_0^2}{c^2}\phi \quad where\ \nu_0 = \frac{m_0c^2}{h} \tag{2}$$

for a spinless wavicle moving along the z axis with group velocity v_g, and then he abandoned the whole theoretical venue [91] p. 19. Construction of wave packets as superposed plane waves is discussed in [93] p.5ff. Since even light cannot be viewed as spinless/unpolarized in general, Klein-Gordon and the like equations were usually regarded as not quite appropriate for depicting spinning wavicles. Concise presentation of Klein-Gordon equation can be found in [94], [95], and Green's function for Klein-Gordon equation is discussed in [93] p. 296f.

Yet if the inner frequency would be treated as an imaginary interval, then the eq. (2) could be identified with the Klein-Gordon equation, in which case the imaginary unit would have to be regarded as quasigeometric operator; this fact suggests that in a multispatial setting a complex equivalent of the eq. (2) could lead to a viable alternative. In fact, 6D Klein-Gordon equation was investigated even for particles with spin [96]. The eq. (2) could be viewed as a subtle hint indicating that previous failures under the SSR paradigm might become worthy of reconsideration when reevaluated afresh, under the MSR paradigm though.

6. DE BROGLIE PERSUADED TO WATER-DOWN HIS DOUBLE SOLUTION IDEA

Shortly after de Broglie proposed apparent wave-particle duality, his idea was confirmed in experiments and was further developed mathematically by Schrödinger whose equation became workhorse of quantum mechanical applications. Why then – despite of great practical value of the Schrödinger equation – de Broglie and some of his successors consider the interpretation of the wave-particle duality as still an open question today? Even though valid, the – operationally accepted but perhaps mathematically (here: structurally) misunderstood – wave-particle duality raised questions regarding the character of the duality and the physical nature of his idea of double solution, even for de Broglie himself.

Let us review the mathematical obstacles that led de Broglie to abandon his original idea of interpreting the wave-particle duality through the concept of double solution, which he then reduced/downsized to the concept of 'pilot wave' that was further advanced by Bohm. The pilot field could also be interpreted as the Evans-Vigier field $\mathbf{B}^{(3)}$ in which case one may replace the usual term wave-particle with the notion 'field-particle' [97].

According to Majorana, light quanta are nothing but an intuitive aspect of a physical entity that in other cases manifests itself as a wave [98], which could become virtually identified with a multispatial approach even though he did not repel the unspoken SSR paradigm. In my opinion, the Majorana formulation of Maxwell's equations [99] is perhaps their most elegant expression that is also suggestive of conceptual necessity to shift the SSR paradigm to the MSR one. The demand for adoption of the MSR paradigm comes not because of apparent formal perfection of the Majorana formulation but rather despite of it, for ultimate perfection would not permit any expansion of the perfectly formulated equations onto any prospective new/better ideas. For their expansion into a conceptually wider, yet heterogeneous domain, seems also necessary.

One objection against the de Broglie-Bohm causal interpretation of quantum mechanics was that the particle does not react dynamically on the wave it is guided by, i.e. there was no reciprocal action of the particle on its guiding wave [100] p. 26, in the interpretation that was favored back then. This objection is valid from the standpoint of the SSR paradigm. But from the viewpoint of the MSR paradigm the lack of interaction between the pilot wave and the particle guided by the wave merely suggests that the pilot wave and its particle could be two distinct views of the very same single underlying entity, namely wavicle. Recall that the de Broglie wavelet φ is an exact localized oscillating solution of the wave equation $\Box\varphi = 0$, which moves like a massive particle even though it is a solution of massless wave equation [101]. For fairly comprehensive discussion of wave-particle duality see [102], [103].

Although the general 1D equation of an abstract mathematical wave traveling along the x direction with constant velocity v is conventionally expressed with second derivatives as

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} \Leftrightarrow f_{xx} = \pm\frac{1}{v^2} f_{tt} \tag{3}$$

it need not be in second derivatives, for if the direction is fixed or chosen then it could read

$$\frac{\partial f}{\partial x} = \pm\frac{1}{v}\frac{\partial f}{\partial t} \Leftrightarrow f_x = \pm\frac{1}{v} f_t \tag{4}$$

when the choice of direction does not matter anymore [104] p. 209f where the minus/plus sign applies to wave traveling in the positive/negative x direction, respectively. Squaring is not necessary when the direction is already predetermined. Even in Bohm's causal interpretation of quantum mechanics a wave equation analogous to (4) can also be written in only first derivatives of the quantum wave function f = Ψ [105] p. 110.

Consider a single event with internal parameters as an exact localized oscillating solution of the Schrödinger wave equation, which Barut called de Broglie wavelet φ moving like a massive particle even though it was solution of massless wave equation, for the wave equation can be derived from the massless wave equation $\Box\varphi = 0$ [101]. In nonrelativistic limit the wavelet of the form $\varphi(x,x_0,\mathbf{v}) = F_\omega(x,t,x_0,\mathbf{v})\exp i(\mathbf{k}\bullet\mathbf{x}-\omega t)$ with $\mathbf{k} = m\mathbf{v}/\hbar$ and $\omega = \Omega+\hbar k^2/2m$, he has concluded that when its internal frequency Ω tends to infinity $\Omega\rightarrow\infty$ then the localization function F approaches δ function, and in the limit $\Omega\rightarrow 0$ F approaches unity and we get plane wave equation that has no longer internal parameters [101]. The wavelet can be interpreted thus as yet another theoretical hint at the apparent need to deploy paired dual reciprocal spaces because single-valued zero and set-valued infinity cannot unambiguously coexist in the very same space, no matter what kind of projection mathematics might offer to accommodate their postulated coexistence. It is because photon of high energy behaves like particle and thus can be localized which fact even mathematicians acknowledge [237]. +@#

Moreover, despite of being clearly of relativistic origin and experimentally confirmed, the de Broglie idea of wave-particle duality either runs contrary to certain tenets of traditional mathematics or may prompt us to consider altering the traditional mathematical approach to differential geometry in particular. This hint refers to the Hilbert's theorem which says that a complete geometric surface with constant negative curvature (i.e. pertaining to hyperbolicity and, as a consequence, also to special relativity) cannot be isometrically immersed in $\mathbb{R}^3$ [106] p. 451ff. Therefore a sphere minus a single point (pertaining to stereographic projection and extended complex plane with a point at infinity [107], [108]) is not a complete surface, in which case it would not be extendable [106] p. 331f. Since isometric transformations preserve inner product and norm [109], deployment of certain paired multispatial structures seems both desirable and unavoidable. Nonetheless, the SSR paradigm itself is not the chief source of this – and many other – conceptual problems encountered in physics. Rather it is because of lack of denumerable operational infinity of field variables in both mathematics and physical sciences, which makes it impossible to accommodate it in the SSR framework, that plagues all hard sciences [110]. Blind acceptance of the unspoken SSR paradigm effectively caused de Broglie to abandon his idea of the underlying wave-particle entity (or wavicle as it is nowadays called) – though not the experimentally confirmed wave-particle duality – and he replaced it with the lesser idea of pilot wave, which he eventually abandoned too in despair, even though it can be defended without defying the SSR paradigm, as Bohm later showed.

The mathematically perplexing double solution of original wave mechanics calls for brief review that shall be aimed at showing that if the former SSR paradigm were shifted to the MSR paradigm then all the allegedly fatal mathematical contradictions that once hampered de Broglie's reasonings would become nonissues, provided they were recast under auspices of the multispatial umbrella of the MSR paradigm deployed with quite unrestricted division by zero, whose lack at that time virtually pushed the aforementioned de Broglie's ideas into oblivion. Now let us briefly reanalyze some of his own documented or just reported doubts, from the multispatial point of view of the new synthetic mathematics, though.

De Broglie's most intuitive physical issue with his conceptual understanding of the wave-particle duality was that the wave amplitude diminishes as the wave spreads whereas its associated particle must have a kind of permanence [111] pp. 226f,243f. Some authors even went so far as to claim that matter is what is conserved (or preserved) [112]. Just contrasting the spreading of one magnitude on one hand with the diminishing of its associated magnitude on the other hand, is indicative of clearly reciprocal relationship between these magnitudes.

Since reciprocity is the main feature of paired spaces (each equipped with its own native homogeneous basis) within a multispatial hyperspace, one can expect that under the MSR paradigm this very dilemma de Broglie encountered can become nonissue. The allegedly problematic – though realistic – feature of the wave-particle duality is perhaps the first indication that neither wave mechanics nor quantum mechanics nor its anticipated transition into a string/M- theory can be successfully aided by the previously developed mathematics that is entrenched under the theoretical umbrella of the unspoken SSR paradigm.

7. NONINVERTIBLE SINGULARITY DEALT BLOW TO DOUBLE SOLUTION

At the beginning of his work Schrödinger put forward the idea that particle is a group of waves i.e. a wave packet. Nevertheless, de Broglie realized that if it really were so then in an experiment like that of diffraction of an electron by a crystal, the wave packet could be completely dispersed and destroyed [113]. Yet the electron is obviously stable. Hence perhaps the particle is not exactly just the classical wave packet as the latter was envisaged back then, but something else, requiring a different depiction. I intend to show that a different kind of mathematics is needed to reconcile the two conflicting pictures of the wavicle.

De Broglie abstractly defined particle as a manifestation of energy or momentum [114]. Note the word 'or' present in his definition. This can be yet another subtle hint at the need for its multispatial representation. Hence the nature of so-defined single wave-particle entity (wavicle) needed further clarification. For fairly simple introduction to the topic see [115].

De Broglie envisioned his double solution theory, in which the quantum mechanical wave function Ψ conventionally associated with probabilistic interpretation, is accompanied by a certain, more physical than Ψ, wave function u() [111] p. 89ff,209ff. The theory envisaged particle as just a singularity (in an extended wave phenomenon) [111] pp. 263,291, for the principle of double solution asserted that to every continuous solution of the equation of propagation of wave mechanics (ibid) there must correspond a certain singularity solution where the coefficients a, f, are the respective amplitudes, and φ denotes phase common to both waves. For concise explanation of the phase harmony see [116], [117]. Here $\hbar$ = h/2π is the reduced Planck's constant and h is the regular Planck constant [111] p. 99f and i is the imaginary unit. The u wave cannot be identified with the singularity depicting the particle [111] p.220. Yet the intrinsic mass of the particle could not be – in this framework – an internal property of the particle but would result from its interaction with the ensemble of particles of the same type in the [presumably hidden beneath the regular quantum level] subquantic medium endowed with a huge quantity of hidden energy [118], he claimed.

Presumed presence of a hidden subquantic level is only of an interpretational significance that is not really necessary for the deductive reasonings to be made under the MSR paradigm, even though the prospective presence of complementary hidden variables is an inherent feature of differential calculus itself [119]. Unlike the supplementary hidden variables, which are viewed as underlying nonlocal features of quantum mechanics in de Broglie-Bohm theory [120], the complementary hidden variables were not offered for making the orthodox quantum mechanics deterministic, but they emerged naturally from differential extensions suggested during reconciliation and explanation of some formerly unanticipated strange experimental results. Note that during de Broglie lifetime only the SSR paradigm was adhered to.

When he constructed Green's function of the wave equations he envisioned for the "more physical" – i.e. less artificial for him than Ψ() – wavefunction u(), he obtained the expression

$$u(M) = \sum_i \frac{\epsilon \Psi_i^*(Q) \Psi_i(M)}{k^2 - k_i^2} \tag{5}$$

where the singularity 1/r exists at the point Q defined in δ(M-Q) as the singular Dirac delta function relative to the singularity point Q where the wave equation cannot be satisfied – see [111] p. 22. Recall that from Green's formula written in complex notation Cauchy's integral theorem (CIT) follows [121].

The operational meaning of the Cauchy's integral formula (CIF):

$$\oint_C \frac{f(z)dz}{z - z_0} = \oint_{(C - z_0)} \frac{f(z)dz}{z - z_0} + \oint_{C(z_0)} \frac{f(z)dz}{z - z_0} \Longrightarrow \{SSR\} \oslash \{MSR\} \tag{6}$$

means separation of the contour C enclosing the singular point/pole z_0. Here is the contour without the point z_0, and is the extracted and separated contour about the point z_0 alone [122], [123], [124] or illustrated in [125]. As usual f(z) is a function of complex variable z. The CIF effectively means separation of the primary domain within the contour C from the extracted domain surrounding the singular point z_0. CIF is the counterpart of the CIT with reciprocal point/singularity virtually demanding separation of the singular point. Hence the nonsingular contour corresponding to the SSR setting and is the extracted contour enclosing the singular point which is turned into paired reciprocal space in the MSR setting. The sign ⃠ means that the singular MSR part of CIF is still evaluated in the SSR setting. The need for shift from the SSR paradigm to the MSR paradigm is thus implicit in the CIF. The MSR paradigm does not defy complex analysis but makes it operationally viable.

In other words: if this separation facilitated by CIF would be abstracted into a higher-dimensional hypercomplex domain then it could mean associating (or pairing) a secondary dual reciprocal space with the primary hypercomplex space, in which case the imaginary unit i becomes an algebraic interspatial operator denoted here by î, or the geometric multispatial operator whose value is: |î| = || = |i| = √(-1). But scalar inner products are equal unity <i|i> = <j|j> =…= 1 [126] p.2. In order to pursue and implement the hint implicit in the CIF a shift from the SSR paradigm to the MSR paradigm is necessary for consistency of operations.

One of Green's identities, which is also known as integration by parts formula, splits triple integral with scalar product of two differential nabla operators into a circular double integral minus yet another triple integral with squared nabla operator (Laplacian) [127]. Note that differential manifold is a space without natural coordinate system, which must be defined locally by mappings [128] and so it is not independent of the espoused paradigm. In the sense, avoidance of relying on differential manifolds is oftentimes prudent and justifiable. By the same token, however, too overconfident reliance on abstract differential manifolds and on set-theoretical mappings can be detrimental to conceptual thinking. Thus, some differential operators may need to be redefined afresh under auspices of the MSR paradigm.

One can see that in order to further pursue this venue of inquiry, i.e. to redeploy the eq. (5) according to (6) in a higher-dimensional hypercomplex setting, we would have to substantially redefine some of the differential operators, which is tremendously important and pretty extensive yet rather tangential – with respect to our present considerations – topic. For complex matrix representation of quaternions see [129] p.36f, the converse of which could be perceived also as an expansion of the usual 2D complex domain into 4D quaternionic or higher-dimensional hypercomplex domain.

For the time being let us continue our analysis in the traditional setting just as de Broglie did. In stationary state the function u() must have frequency equal to that of the stationary state considered, i.e. k must be equal to one of the k_i. The coefficient of $\Psi_i(M)$ is infinite except when $\Psi_i(Q) = 0$ [111] p. 219f. For concise expositions of Green's functions see [130-132]; an extensive discussion of Green's functions can be found in [133] p. 2ff; and the related issues of symmetry and reciprocity of Green's functions are concisely presented in [134].

The eq. (5) is essential for understanding the de Broglie's own doubts arising from his interpreting it under the unspoken but still silently reigning supreme SSR paradigm.

Since the singularity is situated at a point where $\Psi_i = 0$, which – according to de Broglie – comes into fatal conflict with the conception of wavefunction u() having a mathematical singularity for the wave should be found where the Ψ wave would be zero that is exactly at the point where, according to the statistical meaning of $|\Psi_i|^2$, it should be impossible to find it [111]

p. 220. Although de Broglie reasoned quite correctly – insofar as quantum mechanics is concerned – for the probability of the particle's presence is zero where the particle was supposed to be found according to Born's probability interpretation of the wavefunction Ψ. Yet de Broglie's mathematical inference that led him to the conclusion that the probabilistic interpretation constitutes a fatal conflict with his idea of wave-particle duality is not really admissible, because it is correct only under the SSR paradigm. Had he made his inference under the MSR paradigm, he would have concluded that probability of finding the particle is equal 1, because now $\Psi\Psi^* = 0\cdot\infty = 1$ and therefore finding the particle (or the wave) in one of the paired dual reciprocal spaces would be sure. Notice that it was not de Broglie's fault but rather the abhorrent infantilism of the former pure mathematics, which still operates under the SSR paradigm, that he blindly entrusted mathematical interpretation of his ideas.

Thus SSR-compliant reasonings forced de Broglie to abandon his idea of the physical wave u representing a point-singularity like the usual [radial] scalar magnitude 1/r that conventionally represents the scalar potential. Note that almost every topic ever considered in mathematical and physical sciences until now was evaluated under the umbrella of (and/or deduced from) the SSR paradigm. Nevertheless, what de Broglie regarded as fatal blow to his double solution idea can become gateway to a new physics in the spirit of de Broglie (not just in the Bohm's continuation of de Broglie's pilot wave theory) if it is recast under the MSR paradigm in conjunction with unrestricted division by zero. The latter allows us to (meaningfully) operate on infinities once they are depicted in ensembles or hyperspaces, i.e. quasispatial structures (as opposed to just sets) of paired dual reciprocal spaces. For the allegedly fatal conflict that virtually doomed his physical interpretation of the wavicle is only fatal blow dealt only to the mathematics that explicitly prohibited recognition of the division by zero. Were it not for the prohibition, his particle could be perfectly localized within the wavicle, provided it would be recast in the MSR setting, i.e. in a pair of dual reciprocal spaces.

8. WAVE-PARTICLE DUALITY CAST IN PAIRED DUAL RECIPROCAL SPACES

It is known that the general solution to a linear nonhomogeneous differential equation [which – in my opinion formed under the MSR paradigm – would be required for the wave associated/paired with the particle of the single wavicle] can be decomposed into a particular solution of nonhomogeneous equation plus a general solution of the associated homogeneous equation; hence Green's functions provide a method for finding particular nonhomogeneous solutions [135]. Therefore in my opinion, the Green's function (5) supplies theoretical hint suggesting that the wave function Ψ and its conjugate Ψ* should be placed in separate though associated (i.e. somehow interdependent as paired) distinct spaces, and consequently also the conjugation is not the essentially additive conjugation (in group-theoretical parlance that is effectively used in traditional complex analysis) but a (prospective) multiplicative interspatial conjugation yet to be formally defined and comprehensively discussed elsewhere. We might evade this conclusion and keep on deluding ourselves, but the abstract mathematics from which it emerged, is unescapable, even though inconvenient to accept under the old SSR paradigm. For it virtually prompts us to dump the SSR paradigm into trash bin of history.

Under the MSR paradigm the traditional probabilistic evaluation of $|\Psi_i|^2 \Leftrightarrow \Psi\ \Psi^* = 1$ means that if one of the wave functions Ψ (or its conjugate Ψ*) equals zero, then the other function: Ψ* (or Ψ) should be equal to infinity for it would yield $\Psi\ \Psi^* = 0\cdot\ \infty = 1$ because zero and infinity are mutually reciprocal according to realistic understanding of spacetime [46], [136], that relies on Euler [137]; see [45] for actual implementation of the reciprocity. For operationally viable infinity cannot be directly represented within the very same space with homogeneous basis (whether algebraic or geometric) of the primary space in which the primary zero is represented and thus can be located and depicted [44], [45], [138]. The MSR paradigm complies thus with Born's probabilistic interpretation of quantum mechanics.

Hence under the MSR paradigm, the zero probability would be turned into infinite probability though within the dual reciprocal space that is paired to the apace that is assumed as the primary space, in which case the eq. (5) would actually ensure "absolute/indisputable" presence of the particle of the wave-particle duality pair with probability equal to one without ever defying the probabilistic interpretation. The MSR paradigm does not call for rejecting the orthodox Copenhagen interpretation of quantum mechanics because perception of observables depends on their representation (i.e. the view that allows them to be detected) and the MSR alters only the way in which what is being observed is mathematically represented.

9. LACK OF OPERATIONAL INFINITY DOOMED THE DOUBLE SOLUTION

If eq. (5) is evaluated in a single space under the unspoken and thus uncontested SSR paradigm, then one may be inclined to use the unconventional division by zero [138] (where zero expresses impossibility [139]), instead of the conventional Eulerian division by zero (0/0)=1 that stems from Eulerian tradition [46], [137]. Yet what is impossible in a single space may well be doable in paired spaces, which is clear when bases are shown explicitly. It is known that basis of a vector space is not a unique property of the space [133] p. 36, but a property of representation of geometric objects immersed in the space. Hence one's predilection for using the unconventional division by zero may be recognized as implicit preference for deployment of the SSR paradigm, in which case no homespace is designated for the operational infinity to dwell in.

While the conventional division by zero implies that zero and infinity are mutually reciprocal: $(1/0) = \infty$, the unconventional division by zero assumes that $(f(z)/0) = 0$ for any complex function $f(z)$ and thus avoids infinity as an operational algebraic entity altogether [140-146], and as such it could be helpful in absence of operationally viable infinity, i.e. under the old unspoken SSR paradigm.

These two approaches to division by zero should not be seen as competing for superiority because they are actually complementary – see [44], [45], [138]. Hence the choice of one over the other is not a matter of their validity but rather that of selection (even if implicit) of the underlying operational paradigm to be deployed for algebraic operations.

That is why various authors have shown numerous examples where the unconventional division by zero may appear as coming handy even though some of its results are questionable when reevaluated under the MSR paradigm. It is because of the unspoken SSR paradigm that maintained almost statutory disregard for infinity – both operational and structural – with just a few exceptions in analysis. Since the unconventional division by zero is not implicitly shown as related to any particular algebraic basis, it amounts to effective mapping of infinity to zero, which erases the conceptual distinction between these two entities that might be important for some applications even in the SSR setting [238].

If the notion of infinity is rejected *a priori* as mathematically (i.e. both algebraically and structurally/geometrically) unrecognizable entity, then use of the unconventional division by zero may appear as viable an option, which can be supported by numerous examples. But the most constructive role of the unconditional division by zero is that it could also be viewed as indicating the need to apply multispatial approach to the abstract mathematical (and physical) reality, in which case at least two distinct and different homogeneous algebraic (or geometric) bases are necessary [138]. The utility of the unconventional division by zero resembles that of sentences in 2-valued logic: for a sentence A it is not that if A is any formula whatsoever then either A or its negation ¬A is true (under the given interpretation) but only if it is an open formula and if its universal closure is true or false [147].

To ascribe universal validity to the unconventional division by zero can lead to conceptual mistakes, often made quite inadvertently. I do not see how the unconventional division by zero can be helpful for evaluation of the eq. (5). In other words: there is not just one rule for division by zero that is absolutely applicable everywhere. If for some reasons one cannot allocate unambiguously a (geometric/structural) homespace for the operational infinity to dwell in, then the unconventional division by zero may appear as desirable option; otherwise the conventional division by zero may be preferred.

Recall that placing both zero and infinity within the same space is conceptually quite inadmissible because unlike the single-valued operational zero, the reciprocal to it infinity is actually a set-valued operational entity, and thus also structurally the infinity must be a set-valued entity in need of its own space, wherein the infinity could spread out. De Broglie, as virtually everyone else back then, disregarded infinity as operationally feasible entity, not to mention its geometrically viable structure, and he paid the price for the virtual rejection of infinity by failure of his experimentally confirmed ideas. The failed localization of particle in wave mechanics is vivid example of fiasco of the SSR paradigm and of the unconventional division by zero. Acceptance of mathematical prejudices against infinity is detrimental to physics. De Broglie's interpretation of the eq. (5) reveals insanity of the unwarranted prohibition of division by zero by decree in particular, and blind acceptance of the SSR paradigm in general, by leading him to wrongly conclude impossibility of localization of the particle within the wavicle.

Mathematics is mainly about forms and operations, and thus is truthless, but its objects must not only be consistent but also realistic, i.e. procedurally operational and structurally constructible. Yet presence of realistic operations and existence of constructible structures for the operations to be performed on the structures should be confirmed by experimental results. Mathematical truths cannot be established by abstract mathematical means alone. Yet the unconventional division by zero can reveal where the mathematical truth is about to vanquish due to unsubstantiated existential postulates or arbitrarily decreed operations. Mathematics must not be forced into submission by decrees, for enforcing nonsenses can backfire by producing faulty/contradictory conclusions, the acceptance of which can lead to failures.

10. UNCONVENTIONAL DIVISION BY ZERO CAN INFER FAULTY MAPPINGS

Notice that the above reasoning, conducted under the MSR paradigm that vindicates mathematically the de Broglie's ideas, implies that the operational/algebraic infinity hosted in a space Q transits into zero housed in space P (i.e. within the primary space associated and paired with Q) according to the procedure chain: $0_p \cdot \infty = 1 \Longrightarrow 0_p = 1/\infty \Longrightarrow 0_p \rightarrow 0_q$ where the original/primary zero 0_p housed in the primary space P transits into the inverse/associated zero $0_q = 1/\infty$ which is housed in the associated space Q that is dual reciprocal to the primary space P. Each of these spaces is equipped with their own native homogeneous basis indicated by the respective subscript, P with its native basis p and Q with its native basis q.

Under the – still reigning supreme even though usually unspoken – former SSR paradigm the whole idea of infinity was treated as a merely unspecific philosophical concept whereas under the MSR paradigm the traditionally algebraic/operational infinity gained also definite geometric/structural representation as a gateway that leads to a dual reciprocal space from the primary space in which the primary zero is housed [44], [45], while retaining its Eulerian algebraic flavor as the operational inverse of zero [46], [137].

The informal implication $0 \cdot \infty = 1 \Longrightarrow\ 0_p = 0_p \cdot 1$ with $\infty = 1/0$ could also be written as

$$0_p = 0_p \cdot (1) = 0_p \cdot (\infty_q \cdot 0_p) = 0_p \cdot 1_q \tag{7}$$

where we see zero and infinity being represented in two distinct reciprocal algebraic bases p and q native to two different paired spaces, namely the primary space P and its dual reciprocal space Q, respectively, see [138] for examples and [46], [136], for more traditional justification of the generic relation $0 \cdot \infty = 1$ employed above. Note that $1_p = 1_q = 1$ is implicit there.

The dot denotes scalar multispatial multiplication involving distinct bases. While the numbers 0 and 1 can be represented in any basis in either the space P or any other space, including Q, the eq. (7) indicates that the number called infinity (as operational inverse of zero – needs a distinct extra space Q that is paired with the primary space P. The paired space Q should be equipped with a distinct and conceptually different algebraic (or geometric) basis q in order for the infinity to be unambiguously represented. This is done because zero is single-valued operational algebraic entity whereas the infinity is set-valued operational entity.

Therefore, even though the equations $0=1/\infty$ and $\infty=1/0$ are often accepted at their face value, the Riemann sphere and the mapping called stereographic projection celebrated in complex analysis [148] and used to explain infinity as a structurally viable entity, does not really make conceptual sense, for the mapping is of one-to-many and thus it could not be addition of a single point at infinity. One-to-many mapping is trivial but its reverse, namely many-to-one mapping is a simplistic sink at best. For the North pole cannot be mapped onto equatorial plane [149]. This nonsense called stereographic mapping is just pseudogeometric oversimplification that hints again at the need for deployment of paired dual reciprocal spaces.

The eq. (yh) shows that the transition between paired spaces remains reciprocal and so it preserves truly multiplicative inversion (as opposed to merely additive reversion of the phase/argument combined with multiplicative inverse of the modulus, both of which are tacitly combined in complex analysis where inverse of complex number z is mindlessly rendered as $w = z^{-1}$ which can sometimes lead to confusing misconceptions. Operationally viable implementation of realistic conversions between zero and infinity performed on the fly in transit between paired dual reciprocal spaces are shown by examples in [44], [45], [138]. Again, I am not criticizing the particular textbook, for this approach is employed almost everywhere. Yet since pairing of spaces is fairly simple, I am wondering whether there is a deeper reason for this kind of tacit avoidance of – mathematically obvious truth – when these topics are presented to otherwise bright adepts of mathematical complex analysis.

11. DOUBLE SOLUTION VIRTUALLY DEMANDS THE MSR PARADIGM

Contrary to what de Broglie interpreted as alleged impossibility of presence of the particle associated with the guiding it wave is mere an impossibility of coexistence of the particle with the wave that is associated with the particle if the two distinct views (particle and wave) are supposed to be represented within the very same space i.e. geometric structure. Hence the allegedly fatal mathematical conflict from the standpoint of the former SSR paradigm became nonissue from the point of view of the MSR paradigm, which was not known to de Broglie or his contemporaries, for I came up with the multispatial idea after reconciling formerly unanticipated results of some experiments [82-84], [150-152].

But this conclusion also implies that for the particle to actually exist it must be housed in a separate space that is dual reciprocal to the space in which the wave (associated with the particle) dwells. The de Broglie idea of wave-particle duality was conceptually admissible even though perhaps explained not quite right in terms of physics, but the former mathematics, with its unwarranted prohibition on division by zero, was wrong and thus he was incapable of giving proper interpretation of the eq. (5). This particular conceptual fault of former mathematics is due to the fact that the infamous prohibition on division by zero effectively denied a space wherein the infinity (as operational entity inverse to zero) could be housed or just hosted and so be explicitly represented, both structurally and procedurally/operationally.

Hence the eq. (5) virtually hints at the necessity of presence of paired dual reciprocal spaces within multispatial hyperspace. Note that under the MSR paradigm the imaginary unit becomes an interspatial operator and thus the complex conjugate wave function Ψ^* must be relegated to a distinct space, whereas the insistence on unrestricted division by zero implies that the extra space shall be dual reciprocal space with respect to the primary space, which should be equipped with definitely reciprocal native basis.

To be sure we got it right: Mathematics does not lie by itself, but faulty interpretations of mathematical equations and formulas can lead to drawing of false conclusions even if they are drawn quite inadvertently – see [81], [84], for instance.

Although it is well-known that multiplication of vectors by imaginary unit means rotation by 90° – compare [153-156], De Podesta wrote (in the context of quantum mechanics) that the mathematical terms ‘real’ and ‘imaginary’ can be used to describe many two-component quantities, but the terms do not refer to the ‘existence’ (ontology) of the components of the wave function [157]. That is why I treat imaginary unit also as the value of interspatial operators: $|\hat{i}| = || = |i| = \sqrt{(-1)}$. Hence the algebraic and geometric interspatial operators $\hat{i}$ and, respectively, can signify transition between spaces, not only a rotation. Even in a single space the action of imaginary unit can be equivalent to differentiation, which yields tangent magnitudes, even in its usual rendition as rotation by 90° in 2D plane.

Green’s functions are not aberrant. The eq. (5) defies the possibility of existence of the particle under the SSR paradigm, which equates the whole mathematical universe to a single space that is usually identified with a single set, even if it is an inhomogeneous set. Therefore, under the MSR paradigm the eq. (5) defies only the coexistence of these two discrepant representations of the underlying hypothetical entity depicted by the mutually conjugate wavefunctions Ψ and Ψ^*, which can be called wave-particle or wavicle, within the very same geometric space or a quasispatial structure, in general.

The MSR paradigm suggests that the particle of the wave-particle dual pair (or wavicle) could be found in a dual reciprocal space that is associated with the primary space in which the wavefunction Ψ is defined. The singularity point (i.e. infinity), which is regarded by de Broglie as representing the particle, cannot dwell in the very same space where the wave of the wavicle entity lives, as it was envisioned in his double solution theory.

Hence the often used assertion under the SSR paradigm that fields and matter are both undulatory and simultaneously particulate [86] p. 38, can be fairly well rephrased under the MSR paradigm as indicating that wave and particle are merely two mathematically distinct and conceptually different views focused on the same wavicle representing both the wave and particle within a multispatial hyperspace, whose multiple dimensions are bundled together in 3-tuples (when they are cast in orthogonal 3D homogeneous bases) within several 3D spaces. If cast within 4D heterogeneous bases, which seem to partly overlap, however, their formal representations would become much more convoluted because these form either (3+1)D or (1+3)D partly overlapping quasigeometric spatial structures that correspond to 4D spacetime (with the usual signature – – –+) and 4D timespace (with reverse signature +++–), respectively – compare [33], [36]. This split is puzzling orthodox mathematicians [158]. Note that one can obtain similarly intriguing split with the use of Cremona transformations [159]. The spatial overlapping is not paradoxical if it is considered from the standpoint of the MSR paradigm. Note that the misguided traditional acceptance of the previously unspoken (and thus never contested) SSR paradigm is just an oversimplifying presumption. It is not based on any nonpostulative principles.

Some paradoxes created by strict implications are presented in [160]. My preference for using syntheses over strict implications in conjunction with formalized proofs is rooted in (and stems from) the fact that our formal knowledge is always somewhat incomplete, even if defensible by deductions or even confirmed by experimental evidence. The best example is the unquestioned uncertainty principle, which is discussed in reference to laws of physics in [161]. Although I am not challenging the validity nor substance of the uncertainty principle, I have shown [ja] that the scope of the principle can be extended onto multispatial structures where it attains slightly different meaning, which in turn implies that string theories were not really a failure of the "stringy" ideas of their creators – as some authors claimed – but they succumbed due to conceptual inadequacy of former mathematics. While synthetic methods or fuzzy sets, points, relations and functions [162] may not appeal to "strictarians", we should not ground our mathematical evaluations only on the present, often conceptually very incapacitated or "constricted", incomplete knowledge.

Relying too strictly on set-theoretical approach to spatiality, dimensionality and on the notion of infinity in particular – as these concepts were sometimes (mis-)understood by the great minds in the past – may sound indeed scientific, but it all too often resulted in generating veiled misconceptions, even if inadvertently. Similarly, some reductions of mathematical reasonings to mere formalized logical formulations spurred the overall sense of completeness but actually they defeated most honest attempts at understanding of abstract mathematics and especially some of its missing links to reality – such as operationally viable notion of infinity.

Nonetheless, I would recommend a review of some ideas of those theorizing past sages yet with adequate meanings of the notions they used. For example: if the statement that "We thus see that the problems of infinity and continuity have no essential connection with quantity, but are due […] to characteristics depending upon number and order." [163] p. 194, is taken at its face value, then its meaning can become profound, provided that we accept the fact that numbers are entities resulting from algebraic operations and that magnitudes are formed (or perhaps just conceived) within those geometric or quasigeometric structures that correspond to the operational procedures to be performed over those structures.

In the latter case, we arrive at the new synthetic approach to mathematics I was compelled to propose, whether we want it or not. The issue of ordering and the related problem usually introduced by the axiom of choice are discussed also in [164]. If we associate the operational infinity with zero – via mutually reciprocal relation – then the formerly abhorred infinity becomes connected with all the other numbers as well – compare [44-46], [136].

By disallowing existential postulates, which can amount to (often quite inadvertent) creation of nonsenses, synthetic mathematics dispels "rigorous" yet arbitrary reasonings in mathematics, which were based upon conceptually foggy (or perhaps ethereal, if you will) philosophical notions. Under auspices of the MSR paradigm mathematical infinity is both: constructible structural/geometric entity as well as viable operational/algebraic concept.

12. REVIEW OF NUMBER SYSTEMS SUITABLE FOR HYPERSPACES

Pursuing the principle of synthetic approach to sciences which says that mathematical notions should not be settled/grounded only in mathematics or physics alone, one may try to determine them outside their proper sciences. Selection of number system suitable for representing mathematical objects could take cues from philosophy, because it involves infinity. Nicolas of Cusa did not say that infinite numbers actually do exist or that geometric figures extend to infinity, but he extrapolated their properties to infinity, which he viewed as a limit [165] and "unattainable goal" of all knowledge [166].

According to him finite line is divisible; infinite line is not; and the infinite, in which the largest and the smallest fall together, has no parts [167]. He also believed that the actual infinity is unity [168], which is akin to the realization that. His ideas influenced my search for meanings of infinity. Although his views are usually interpreted either allegorically or in philosophical terms, his perception of infinity is close to its mathematical understanding. For mathematical infinity is relative not absolute [169]. Since I am not looking for an inspiration anymore, I will skip review of pertinent philosophical ideas, many of which I have already discussed in some of my previous papers and return to informal inquiry in operational/algebraic and structural/geometric terms.

Since paired dual reciprocal 4D quasispatial structures should employ heterogeneous (3+1)D or (1+3)D bases, one might attempt to declare the apparently 8D octonions as the numbers that could best represent such a pair because of their nonassociativity [126] p. 11ff. Since both gravity and noncommutative geometry are embedded as a pair of quaternions in octonions [183] p. 308f and because of octonionic quantum gravity [183] p. 339f, octonions are well predisposed for representing realistic multispatial structures. Also the fact that mass can be viewed as an extra spacelike component of a higher-dimensional vector [170] is exactly compatible with pairing of dual reciprocal spaces speaks in favor of octonions. However, octonions, which have been developed under auspices of the previous SSR paradigm, are not yet quite ready to be accomodated under the – not fully understood yet– MSR paradigm.

It is also tempting to assume that focusing on biquaternions or perhaps – for the time being – concentrating on quaternions as the numbers that could support the preliminary model for multispatial structures could be the next best choice. However, since all hypercomplex numbers and their abstract algebraic structures emerged under the SSR paradigm, I cannot rely on them in MSR setting; neither on quaternions whose physical applications [171-173], are still somewhat clumsy nor on biquaternions as these used to be algebraically presented – see [174], [175], no matter how appealing these may seem to be.

The fact that some tend to apply hypercomplex numbers to physical applications is very encouraging indeed, but their applicability does not necessarily ensure entirely acceptable representations. Since in the SSR setting spaces are interchangeable with mere sets, let us focus on less controversial representations. For quaternions basics see [176] and for their geometric compositions see [177]. Recall that real quaternions form 4D real vector space [129] p. 9, and therefore, from the point of view of the MSR paradigm, they could also be considered as paired dual reciprocal spaces, rather than as just sets. The 4D quasispatial structures, which could be deduced above from functional de Broglie/Einstein relationships, are similar to those obtained in [45], [44] from relativistic evalations of differential operators.

One might desire an infinitesimal transformation of coordinates and so pursue a concept analogous to Lie derivative. Lie derivative/differential of a tensor has tensor character [178] – compare also group of Lie transformations [179]. However, such a prospective interspatial differential would have to be somehow synchronized as it needs to affect the same object perceived in some two dual reciprocal spaces, which feature is not yet fully developed.
Although hypercomplex number systems seem preferable for representing algebraic properties of objects depicted in multispatial setting, at this time it would be difficult to decide which ones are the most suitable. For now I shall stay with quaternions for Hamilton already recognized that quaternions apparently contain inverse factors (or reciprocal quotients [180].

13. DE BROGLIE'S IDEAS STIFLED BY SINGLE SPACE REALITY PARADIGM

De Broglie realized that a particle's momentum, and hence its scalar velocity (i.e. speed) v, is inversely proportional to the wavelength λ and directly proportional to frequency ν

$$v = \nu/\lambda \tag{8}$$

neither of which is directly representable as variable in the usual 3D Euclidean length-based space (LBS) equipped with a native homogeneous 3D algebraic basis denominated in meters. Notice that the relation (8) is of clearly reciprocal nature.

This is because temporal frequency is expressed and measured in inverse seconds (or cycles) and wavelength (i.e. the spatial or length-based frequency) is expressed and measured in inverse meters. Hence neither is wavelength directly representable in the usual geometric space whose algebraic basis is denominated in meters nor can temporal frequency be directly representable in the space whose algebraic basis is denominated in seconds.

Thus, the spatial frequency λ could only vary as an inverse spatial parameter and the temporal frequency ν could only vary as an inverse temporal parameter, if the operational universe of mathematics is equated with a single space as it was conceived in the unspoken (and thus unquestioned) SSR paradigm.

In other words: In the LBS – which is the primary space P here – both the temporal frequency and the spatial frequency (or inverse wavelength) can be represented only indirectly as functional parameters, not as independently varying variables as they were treated.

The reader averse to subtler mathematics may compare [181] p. 139ff for functional explanations of electron waves. By function-based equations I mean those involving "heavy" differential operations whereas functional equations tend to avoid differentials and rely on integrals, except in definitions or prototypes such as the eq. (9) posted below.

In the present paper I shall try to restrict myself to mainly functional reasonings because differentiation performed within multispatial structures of hyperspaces requires significant changes of differential operators, which shall be done elsewhere. Though my approach is not preferable, usually gatekeepers to scientific publications would tend to reject reworking of differential operators under the pretense that there is no need for that. Hence here is the need. The eq. (8) can also be legitimately written in equivalent terms of simple differentials as

$$v = \frac{ds}{dt} = \frac{d\nu}{d\lambda} \tag{9}$$

if the parameters λ,ν, also could be allowed to vary in their own homespaces, each of which should be equipped with a native basis denominated in units capable of direct representation of the parameters turned into variables that can vary quite independently of all the other variables and parameters involved. Notice that every magnitude (be it variable or parameter) can be represented indirectly in any space, sometimes after conversion to units compatible to those in which the native basis of the given space is denominated. But in the latter case the homogeneity of the extended basis (whether it is algebraic or geometric or quasigeometric) may no longer be maintained. The issue is not impossibility of formal representation but the desire to maintain paired homogeneous representations in two distinct and different bases.

Although the eq. (9) is operationally valid, the corresponding to it geometric structure is conceptually flawed from the synthetic point of view in the sense that the parameters – when turned into variables – are homeless, for they have no space in which they could vary quite independently of all the other variables and parameters.

There was nothing wrong with the experimentally confirmed de Broglie's idea of wave-particle duality. The main reason that caused him to relinquish his idea of double solution is inadequate former mathematics and some not quite appropriate reasonings based upon the faulty traditional mathematics.

Neither he nor his detractors who caused him to abandon his original proposal of double solution (which he replaced with watered down idea of the pilot wave that was further developed by Bohm) understood the mathematics underlying the equations he has written but was not always able to properly analyze in conceptual terms.

The theory found itself in a situation analogous to the present perception of string theories, which is sometimes seen by some authors as utter failure, because its champions were unable to overcome their apparently unsurmountable mathematical and conceptual obstacles [182]. But a multispatial extension of uncertainty principle makes the main stringy obstacle a nonissue [45].

If a lesson is to be learned from the unfortunate fate of de Broglie's abandoned ideas, then perhaps we should allow the development of new mathematical ideas rather than to keep on suppressing them while endorsing pointless digging into irrelevant issues of only historical importance but of no significant ingress into advancement of physical sciences. To keep the – experimentally confirmed – ideas of de Broglie afloat one clearly needs the MSR paradigm.

14. SINGLE SPACE REALITY INFERS UNLOCALIZED MASSLESS PARTICLES

Attention: most symbols used in this section follow the standard naming convention of differential geometry where κ,τ, for example, denote curvature and torsion, respectively.
There is no simple equivalent of angular momentum and angular energy because angular momentum, being an axial vector cannot form the space part of a 4-vector [86] p. 13, for it is impossible by fundamental geometry, to produce a polar vector in Euclidean space from a vector product of two polar or two axial vectors [86] p. 35. Hence localization of massless particles is not defined or possible [86] p. 13, which is certainly true under the previous SSR paradigm. For reasonings made under the – unspoken as it were – SSR paradigm preclude depicting photon as pointlike particle because its mass would be infinitely great as Born has already inferred in [181] p. 73.

For reasonings under the – unspoken as it were – SSR paradigm preclude depicting photon as pointlike particle because its mass would be infinitely great as Born inferred in [181] on p. 73. Notice that this particular conclusion seems reasonable under the umbrella of single space reality that is espoused by most scientists and encapsulated in the SSR paradigm, not because pointlike objects are impossible to conceive but because reciprocal of zero, which is infinity [46], cannot be meaningfully depicted within the same space as zero, even though the latter impossibility is not always acknowledged. Besides, evaluations of infinities are not absolute but depend on the algebraic or geometric basis in which these entities are considered. In other words: because there was no place for infinity to be unambiguously allocated in the same space wherein its mutually inverse pointlike objects reside, then it is said that we allegedly cannot depict the pointlike objects as points, which is clearly illogical a reasoning.

Does this Born's conclusion mean that we should forget forever about the quite legitimate desire to achieve such a localization at all or is its alleged impossibility contingent upon some yet unrecognized mathematical features? Recall that nonlocalizability of energy in a region of spacetime is a kind of uncertainty relation [183] p. 243. Yet although reasonable extension of the uncertainty principle, which then turns into an indeterminacy principle, could be implemented in paired dual reciprocal spaces [45], this extension requires deployment of the MSR paradigm. For under the MSR paradigm localization of geometric or quasigeometric objects is not constrained, neither by the particle's mass nor by its simplified depiction as an abstract mathematical point.

Since the velocity is $\mathbf{v} = \frac{d\mathbf{r}}{dt}$ then for unit speed $v = |\mathbf{v}| = |\frac{ds}{dt}| = 1$ the conventional Frenet formulas are used to evaluate the frame (e_1,e_2,e_3) determined by the trihedron (**T,N,B**) that moves along the trajectory curve pointed at by the radius **r**, we get the following equations

$$\sum_n \frac{\partial e_n}{\partial s} = vM \tag{10}$$

where the local speed v is measured along the trajectory curve s and M is the Frenet matrix form of derivatives of three unit vectors: tangent **T**, principal normal **N**, and binormal **B**:

$$\boldsymbol{T}' = \qquad \kappa\boldsymbol{N} \tag{11}$$

$$\boldsymbol{N}' = -\boldsymbol{\kappa}\boldsymbol{T} \qquad \tau\boldsymbol{B} \tag{12}$$

$$\boldsymbol{B}' = \qquad -\boldsymbol{\tau N} \tag{13}$$

with unit speed in 3D Euclidean space. Here κ is the principal normal curvature and τ is the torsion (i.e. the other, binormal curvature) of the curve [184] p. 20f; see also the Frenet matrix with v≠1 on p. 28f. Note that n = 1,2,3 correspond to the sequence indicated by the Frenet trihedron (**T,N,B**) and thus each derivative **T',N',B'**, is determined by single row of the Frenet matrix M. Here ds denotes the differential of arclength along the curved path identified with trajectory and t is the elapsing time parameter. Note that the Frenet matrix is skewsymmetric and thus is indicative of twisting motion of the trihedron that moves along the path curve (and thus can be twisting with the moving trihedron) that the moving along it particle also follows. Generalizing the Frenet matrix for a curve α with respect to any unit vectors **G**(t), **H**(t) perpendicular to the tangent unit vector **T**(t) of the Darboux vector $\boldsymbol{\omega} = \omega_1\boldsymbol{T} + \omega_2\boldsymbol{G} + \omega_3\boldsymbol{H}$ along the path we obtain the following Darboux matrix, compare [184] p. 21, [185] pp. 26ff,52:

$$\boldsymbol{T}' = \qquad \omega_3\boldsymbol{G} \quad -\omega_2\boldsymbol{H} \tag{14}$$

$$\boldsymbol{G}' = \; -\omega_3\boldsymbol{T} \qquad \omega_1\boldsymbol{H} \tag{15}$$

$$\boldsymbol{H}' = \qquad \omega_2\boldsymbol{T} \quad -\omega_1\boldsymbol{G} \tag{16}$$

where **T' = ω×T, G' = ω×G, H' = ω×H,** and **H = T×G.** For our purpose it is also important that the unit vectors of the Frenet trihedron (**T,N,B**) and the Darboux trihedron (**T,G,H**) are clearly associated with speed/velocity (first derivative), curvature (second derivative), and torsion (third derivative) of the function that describes motion of the respective trihedron [186], [184].

Frenet and Darboux matrices suggest need for an extra longitudinal (i.e. nontwisting with the twisting trihedron) diagonal line element, which must be located on a diagonal in order to ensure localization. Yet to be nontwisting, the extra line element must be placed in another space because it cannot sit in the empty/zero-diagonal within the twisting trihedron matrices. For the Frenet and Darboux trihedrons that move along the trajectory are kind of localized but on the path/curve, not at a point, as localization in space is usually understood.

The latter conclusion is easy to understand in electromagnetics, where the magnetic vector **B** – not to be confused with the above binormal unit vector **B** in (12) and (13) that is also conventionally depicted as bold **B** – is entirely transverse according to Maxwell's equations, whereas the electric vector **E** has both transverse and longitudinal parts, the latter depending on the scalar potential φ such that $\mathbf{E}_L = -\nabla\varphi$ while the transverse electric part depends on time derivative of the vector potential – compare [187].

How the arclength parameters s,t, relate to the position/pointer vector **r** that points to the trajectory and is usually expressed in the rectangular coordinates x,y,z, in terms of differential calculus are shown in [188], [189] in very simple differential and geometric setting. The Darboux matrix also hints at the possibility that its prospective further generalization into higher dimensions may require not only structural/synthetic extension (because there seems to be nowhere else to go if the 3D space should preserve the orthogonality of its geometric dimensions) but also an expansion of the range of the Frenet/Darboux operations far beyond their proper domain, for the prospective expansion could be both operational and structural.

While 'extension' usually implies an addition made within the given domain of the operations, 'expansion' conveys the message that the desired addition should come from somewhere on the outside of the domain, because there is nowhere to extend it in 3D space due to its 3D orthogonality. Note that I am not saying that it is impossible to make transition from 3D to 4D operations, but that comparison of the Frenet operational matrix form with the Darboux matrix form suggest that range of the geometric operations could only be reasonably extended at the cost of expanding it into an extra operational domain lying somewhere else, outside of the 3D operational domain in which both Frenet and Darboux frames dwell.

The approach closest to the MSR paradigm is described in [190] where the [elapsed time] derivatives of {**t**,**p**,**b**} are conveniently expressed in terms of general Darboux's vector **ω** as:

$$\boldsymbol{\omega} = \tau\,\mathbf{t} + \lambda\,\mathbf{p} + \kappa\,\mathbf{b} \tag{17}$$

with time derivative d**a**/dt = **ω** × **a** where **a** is any vector rigidly attached to the trihedron moving along the given trajectory curve so that we can write symbolically d/ds = **d**x in [190].

As one can see, neither the Frenet matrix nor the Darboux matrix contains nonzero line element $\sum x_{jj}$ placed along the diagonal interval, the lack of which is the abstract mathematical reason for the aforesaid impossibility of localization (at a point) when only axial elements are available. For zero-line element is either skewsymmetric covariant tensor field or a differential 1-form [191]. Note that physics usually ignores the fact that tensors are not really full-fledged differential operators even though they appear as if they were. For tensors are not determined by the process of taking derivatives of certain other quantities (which is a local process), but rather through evaluation of known quantities at single points [185] p. 239; they are thus "poor man's" differential operators only pretending to appear as if they really were differential operators, which – routinely unmentioned – fact often created confusion.

Hence the most viable formal or abstract mathematical precondition for the possibility of localization would be recasting the inverse magnitudes that appear in the eq. (8) in a separate space whose native basis should be dual reciprocal to the native basis of the LBS and thus permitting presence of a nonzero diagonal line element. Being mutually reciprocal, zero and infinity can be deployed for transfer from one of these paired spaces to the other [45], [44].

Similar conclusion might be reached from concurrence of purely traditional indicators of the necessity of pairing of dual reciprocal spaces. For if wave tensors are complex tensors as it was expounded in [192], and because the longitudinal Evans-Vigier field $\mathbf{B}^{(3)}$ [86], [87], could be interpreted as the de Broglie pilot field [97], then the 3D representation of the $\mathbf{B}^{(3)}$ field in conjunction with the overall complex representation would exceed the usual three dimensions allocated for the wave.

Hence the imaginary unit should have been interpreted as complex interspatial operator relating two paired 3D spaces within a 4D spatial structure because the attempted 6D quasispatial structure would have to be squeezed into a 4D quasispatial structure due to algebraic restrictions virtually imposed on geometric structures by Abel and Galois [33], [36]. Although quasigeometric 6D structures can be unambiguously constructed, higher than 4D realistic (i.e. constructible) single geometric spaces cannot exist if orthogonality of all dimensions has to be maintained.

One cannot construct simple (homogeneous) nD space for n>3 orthogonal dimensions nor a heterogeneous nD space for n > 4. Higher than 4^{th} dimensions require multispatiality in order to be implemented – see ibid.

One can realize by now that this abstract mathematical precondition for localization is comparable to the de Broglie's original proposal of double solution in which the primary quantum-mechanical wave function Ψ is conceived as being associated with the physical wave u whose singularity was supposed to represent the particle of the wavicle. Though not quite equivalent, for there is no compelling reason for the wave to actually guide the particle associated with the wave – as de Broglie envisioned it – or vice versa (i.e. the particle dragging the wave with it), the new synthetic mathematics required to implement the precondition also demands deployment of quite unrestricted algebraic operation of division by zero. The detail that actual geometric process of constructing paired reciprocal spaces is contingent upon unrestricted division by zero is not an option but a must though tacitly ignored in the past.

15. WAVE-PARTICLE DUALITY DEMANDS PAIRED RECIROCAL STRUCTURES

If $\mathfrak{R}$ is a representation cast in 3D Euclidean set-theoretical real numbers space $\mathbb{R}$ – which in geometric context is usually written as $\mathbb{R}^3$ – that represents the inverse mapping r→(1/r) of a scalar interval r which is length-based (i.e. distance), then if set-theoretical space that is equipped with real-valued basis ⟦ R ⟧ maps the scalar interval r onto the point 1/r residing in a set-theoretical space $\mathbb{R}^{-1}$ equipped with real-valued reciprocal basis ⟦ ʁ ⟧ whose value is just the inverse interval, then, since spaces can be identified by their bases, the representation is:

$$\textit{interval in } \mathfrak{R}\ \mathrm{P}([\![\mathrm{R}]\!]\mathrm{r}) \ \coloneqq\coloneqq \textit{point in } \mathfrak{R}\ \mathrm{Q}([\![\text{ʁ} = \tfrac{1}{\mathrm{R}}]\!]\tfrac{1}{\mathrm{r}}) \tag{18}$$

where the double substitution symbol ⩴= indicates here both: the substitution of value of the variable r as well as transition with conversion from the native basis on the left-hand side (LHS) of the primary space P to the native basis associated with the dual reciprocal space Q standing on the RHS, which has foreign basis with respect to that of the primary space on the LHS [45].

The prototype formula (18) is essential for comparisons of intervals r with points described in the form 1/r being reciprocals of intervals, which are quite different kinds of objects, regardless of whether they are algebraic or geometric entities. It also justifies the necessary pairing of primary spaces with their dual reciprocal spaces, i.e. secondary spaces. The unspoken single-space reality (SSR) paradigm appears thus as a consequence of lack of recognition of the prototype formula (18) in former mathematics.

Nevertheless, the prototype formula (18) actually depicts a mutually reciprocal technique. Hence if the given primary space P contains points (or pulses) then its secondary dual reciprocal space Q should contain intervals and vice versa. Hence the prototype formula (18) can also be alternatively called multispatial pairing formula. Traditional abstract pairing of quaternions is shown in [193].

The upside-down symbol ʁ emphasizes the fact that the basis (and the space) standing on the RHS of the formula (18) is reciprocal to the primary space R (and its proper native basis) that stands on the LHS of (18). Notice that the intervals (or line segments) on the LHS of (18) are conceptually different entities than the points on the RHS. Even single-valued numbers cannot always be fully identified with their values, which are mere attributes of the abstract operational entities called numbers, which is contrary to what Georg Cantor presumed [45].

By the way, just looking at the formula (18) that equates an interval with a point, one can see that the traditional mathematics with its quite unwarranted prohibition on division by zero is a breeding place for various conceptual nonsenses, one of which was the tacitly condoned SSR paradigm. For to equate objects of different kind, such as intervals and points, one needs a pair of two distinct and separate spaces, in which the two discrepant entities could be housed. One does not need "heavy" mathematical training to see how illogical the placement of two incompatible objects within the same space was, and then attempting to consider them on the same footing. This inconsistency is just one tip of the proverbial iceberg of nonsenses and contradictions floating freely on the surface of the allegedly "rigorous" reasonings in the abstract sea of notions whose existence was simply postulated in former mathematics, often arbitrarily and sometimes quite inadvertently. That is why the new synthetic approach to mathematics avoids axioms and definitions, which often are just existential postulates in disguise that tacitly introduce objects which either cannot exist at all or cannot coexist with some other objects that are known to exist. Existential postulates are inadmissible.

Abstract geometric/quasigeometric representations are neither functions nor mappings for they only point to (or represent) the very same geometrical object viewed from inside of each of their respective spaces. In former mathematics the inverse mapping $r \rightarrow (1/r)$ is considered as a function that maps r onto its reciprocal, which is not entirely wrong, but it misses the fact that the interval r, which is denominated in meters [m], the yardstick against which lengths are measured, does not really belong in the same space as its inverse 1/r, whose basis is denominated in inversed meters [1/m], i.e. lengthcycles (spatial frequency) or lengthpoints.

This mixup created conceptual confusion in physical sciences, not only because it is wrong but also because it was refused to be recognized as problematic by gatekeepers (i.e. editors and referees) of scientific publications. The refusal to allow rectification of this and some other mathematical flaws suggests that perhaps there were also some other than meritorial reasons for suppressions of valid scientific ideas in the past.

It is acknowledged that the term 'conjugate' in reference to complex function is different than the same term when it is used in refence to complex numbers [194] p. 90. Nevertheless, the reciprocal transformation $w = 1/z$ is not really reciprocal, because it is also additive (with respect to the angle/argument) rather than be doubly multiplicative mapping with respect to both modulus and angle. For complex conjugation reverses phase while inversing modulus 1/r [194] p. 305 (and thus is a product of inversion with respect to unit circle followed by reflection in the real axis [194] p. 306) even though reciprocity often alludes to absolutely multiplicative mapping, especially when it appears as   in this deceptively evasive notation.

Then to cover up the deception it is usually being postulated that an ideal point of infinity is adjoining the complex plane [194] p. 308. Yet the ideal point at infinity ∞ is not really a single-valued point like 0 but actually is a set-valued point. Identifying a single-valued point with an abstract set-valued point is a farce, which the formerly prohibited division by zero was supposed to cover up, rather than to justify. This insane cover-up is indefensible; neither operationally nor structurally. I am not singling out this particular book, for the deception is widespread throughout complex analysis and is pretty common to all traditional presentations of the topic. The issue and its consequences shall be further discussed elsewhere for this and related nonsenses obstruct conceptual reasonings in mathematics and physical sciences.

Writing the functional eq. (8) de Broglie has virtually suggested that the differential function-based eq, (9) equated the usual operational derivative ds/dt of the (respective spatial and temporal) intervals s and t to the structural (i.e. geometric) derivative $d\nu/d\lambda$ (that corresponds to the operational derivative) of inverse intervals, i.e. pointlike variables representing the respective temporal pulse/cycle ν = 1/t and the "spatial" pulse/cycle λ=1/s and for waves in particular λ = 1/k where k = |**k**| is the wavenumber, scalar and vector, respectively. Thus, without denying the operational validity of the eq. (9) we shall be able to represent also an alternate scalar velocity/speed υ structurally in a dual reciprocal space Q per eq. (18):

$$\Re\, \mathrm{P}(⟦\mathrm{R}⟧\mathrm{r})\colon \mathrm{v} = \frac{\mathrm{ds}}{\mathrm{dt}} = \frac{\mathrm{d}\nu}{\mathrm{d}\lambda} \;\Longrightarrow\; \Re\, \mathrm{Q}\left(⟦\text{ʁ} = \frac{1}{\mathrm{R}}⟧\frac{1}{\mathrm{r}}\right)\colon \upsilon = \frac{d\mathcal{V}}{d\Lambda} = \frac{d\mathcal{S}}{d\mathcal{T}} \;\Longrightarrow\; |\mathrm{v}\cdot\upsilon| = 1 \quad (19)$$

where $\mathcal{V}$ = 1/ν is the alternate interval corresponding to temporal frequency ν, Λ = 1/λ is the alternate interval corresponding to the spatial frequency (or wavelength) λ, $\mathcal{S}$ = 1/s is the alternate spatial pulse that corresponds to the trajectory interval s, and $\mathcal{T}$ = 1/t is the pointlike alternate temporal pulse that corresponds to the usual elapsing time interval t. The scalar velocity (speed) v in the primary basis ʀ and the alternate speed υ in the inverse basis ʁ are reciprocal (i.e. multiplicatively inverse). The light dot '·' denotes interspatial scalar multiplication performed across paired dual reciprocal spaces, each of which equipped with homogeneous but quite distinct algebraic or geometric bases.

The magnitudes (i.e. either variables or parameters) highlighted in green signify lengthlike intervals whereas those highlighted in red signify pointlike pulses akin to temporal cycles. The differentials placed in numerators are usually predisposed to represent independently varying magnitudes/variables depicted in the native basis of the given 3D space whereas the magnitudes shown in denominators are more likely to represent either running parameters depicted in a foreign basis of the given 3D space or within a separate 1D furled secondary space associated with the primary 3D space. The color-coding of magnitudes is not essential.

Alternate secondary intervals $\mathcal{V}$, Λ dwell in distinct alternate 3D space Q, which is dual reciprocal to the primary 3D space P that is being paired with the space Q. Alternate 1D pointlike/pulselike magnitudes $\mathcal{S}$, $\mathcal{T}$ correspond to the intervallike magnitudes s and t living in the primary 3D space P, dwell in a distinct 1D space Q that is paired with the 3D space P alternatively represented also as parameters in the secondary/alternate 3D space Q. While pairing of spaces requires either (3+1)D = 4D or (1+3)D = 4D dimensionality, partial overlaying of spaces retains the same dimensionality, (3+1D = 4D ⟶ 4D = (1+3)D. Total overlaying of spaces would be either 1D⟶1D or 3D⟶3D or some other abstract dimensionality.

If the 1D space Q and the 3D space P compose a certain 4D quasispatial structure ST, when it is arranged as a multispatial pair of combined 4D signature (– – –+), then the combined pair resembles spacetime with a 4D heterogeneous basis. Notice that the 1D space Q is not a slice of the 3D space P but a 1D dual reciprocal space to the 3D space P with which it is being paired – see [44], [45], [138] for more details on pairings of multispatial structures and [33], [36] for an abstract introduction to overlapping quasispatial structures.

It is recognized that not only the value of angular momentum but even the value of linear momentum of particle is periodic manifestation too, so that the eq. (9) can be rewritten as

$$\mathrm{v} = \frac{\mathrm{ds}}{\mathrm{dt}} = \frac{\mathrm{d}\nu}{\mathrm{d}\lambda} \;\Longrightarrow\; \mathrm{p} = \frac{2\pi\hbar}{\lambda} \quad (20)$$

which indicates that linear momentum is reciprocal with respect to wavelength λ [195]. Here the Planck constant is rendered as $h = 2\pi\hbar$ which is suggestive of periodicity and $p = |\mathbf{p}|$ where $\mathbf{p} = m\mathbf{v}$ and $v = |\mathbf{v}|$ with m being the mass. For the time being the scalar functionals shall suffice. The formula (20) could also be written in terms of energy and elapsing time interval as

$$E = 2\pi\hbar\nu = \frac{2\pi\hbar}{\tau} \tag{21}$$

where energy E is proportional to frequency ν and reciprocal to elapsing time intervals $\tau = 1/\nu$.

Thus the functionals (9), (20) and (21) conceived under the SSR paradigm comply with the multispatial prototype formula (19) that emerged under the MSR paradigm. Hence the paradigm shift from SSR to MSR needs not be postulated for its possibility is already implicit in formulas developed under the SSR paradigm, even though the shift's feasibility was not always recognized in the past. De Broglie's wave mechanics is thus compatible with the MSR paradigm.

The above functionals can be interpolated and imply also the following relations

$$2\pi\hbar = p\lambda = E\tau \quad\Longrightarrow\quad p\nu = Ek \quad\Longrightarrow\quad p\boldsymbol{\nu} = E\mathbf{k} \tag{22}$$

in accordance with the prototype formula (19). Recall that $k = 1/\lambda$ is the scalar value of the wavenumber vector **k**. Notice that the – purposely chosen as 1D scalar functional relation $p\nu = Ek$ obtained above – should also emerge from a certain prospective 3D vectorial equation

$$p\boldsymbol{\nu} = E\mathbf{k} \tag{23}$$

whose LHS may at first glance appear as not inadmissible, but it could be problematic from the standpoint of the former SSR paradigm, for the – spatial even though formally still unassigned above – algebraic basis of the linear momentum p and the (inverse/reciprocal temporal) vectorial basis required for the frequency $\boldsymbol{\nu}$ are incompatible. Yet under the MSR paradigm it actually is nonissue.

Similar apparent incompatibility affects the RHS of the formula (23) for the temporal algebraic basis of the scalar energy E seems incompatible with the (inverse spatial) vectorial basis associated with the wavenumber vector **k**, if the latter is considered within the usual spatial reference frame (x,y,z). Hence, if the formula (23) is to remain mathematically correct, then it should be evaluated within some two paired 3D structures immersed within a 4D hyperspace, regardless of whether they may be of geometric or quasigeometric nature. In other words: reconciliation of the aforesaid incompatibilities evidently demands shift to the MSR paradigm.

The chain of implications (22) suggests that vectorlike frequency interval should exist in addition to scalarlike temporal frequency, if the equations of wave mechanics should be backward compatible with the functional equations (i.e. equations rendered in terms of functionals/integrals rather than differentials) that de Broglie already came up with, but the mathematics at his disposal was not adequate back then to support his intuitively correct ideas.

In other words: Transition from purely algebraic mathematics to geometric mathematical operations suggested the necessity of the structural relationship (23). Yet if our mathematics should remain operationally correct, the operations must be fully determined with reference to algebraic and geometric bases involved. For synthesized operations should fit the structures they are supposed to operate on and vice versa.

Note that postulating existence of unrealistic (i.e. impossible to construct) geometric or quasigeometric structures does not make sense; neither makes sense proposing operations that could not be unambiguously performed.

However, if the prospective new quasispatial structure imposed by the quasigeometric functional relationship (23) should be actually and realistically constructed – as opposed to just talking about it (which would be "construction in the mind" rather than in reality) or just writing about it (which would be a "construction on paper" that may not be feasible to be unambiguously implemented in the actual physical reality) – so that it could be unmistakably operated on, then there should exist an extra 3D space equipped with its own homogeneous yet separate native basis that is distinct and different than the native basis in which the linear momentum vector is natively (i.e. directly) represented. Even without mentioning bases explicitly, traditional quantum mechanics uses reciprocal representations of position and momentum, though still under the SSR paradigm [196].

Hence the 3D temporal frequency interval vector **v** should have its own space wherein it could dwell and vary quite independently of all the other 3D variables and in particular, varying independently of the varying 3D wavenumber vector **k**.

Some authors claim that de Broglie came up with the idea that any moving body must be accompanied by a nonmaterial wave of wavelength λ = h/p (in absence of a field) was to some extent the consequence of his trying to reconcile relativity and quantum postulate; thus if a mathematical relation is established between first components of two 4-vectors then there is no real reason why we should restrict the relationship to only one component [197] p.31, which can be called unfurling principle. Since 4D spacetime is just a quasispatial structure equipped with (3+1)D heterogenous (3D spatial and 1D temporal) basis, then the unfurling can also be applied to two paired 3D spaces equipped with different bases.

If this line of reasoning is acceptable then from υ = const I can also imply the following

$$d\mathcal{V} = \upsilon d\Lambda \quad \Longrightarrow \quad \int d\mathcal{V} = \int \upsilon d\Lambda \quad \Longrightarrow \quad \mathcal{V} = \upsilon\Lambda + const \tag{24}$$

and, given the implications (19), the functional equation (22) could be further evaluated to

$$\{\mathrm{p}\mathbf{v} = \mathrm{E}\mathbf{k} \;\;\&\;\; \upsilon = \frac{d\mathcal{V}}{d\Lambda} = \frac{d\mathcal{S}}{d\mathcal{T}}\} \quad \Longrightarrow \quad \mathrm{p}\mathbf{v} = \mathrm{mv}\cdot\upsilon\cdot\Lambda = \mathrm{m}\cdot\Lambda = \mathrm{mK} \Longrightarrow \mathrm{mk} \tag{25}$$

where k = 1/λ and K = 1/Λ is the multispatial counterpart of the scalar wavenumber k. Note that the de Broglie's wave accompanying its particle is a phase wave impossible to be separated from the motion of the particle [197] p. 30. To me, this impossibility of separation is a subtle theoretical hint that the wave and particle may just be two diverse pictures shown in quite distinct yet mutually paired dual reciprocal spaces, each of which hosts different view of the same underlying wave-particle entity dubbed wavicle.

Notice that formal inference in the prototype formula (19) proceeds in reverse order:

$$\mathrm{P}\{(\mathrm{ds/dt}) = (\mathrm{d}\nu/\mathrm{d}\lambda)\} \Longrightarrow \mathrm{Q}\{(\mathrm{d}\mathcal{V}/\mathrm{d}\Lambda) = (\mathrm{d}\mathcal{S}/\mathrm{d}\mathcal{T})\} \tag{26}$$

which in an informal notation may be rendered as the following sequence of dual mappings: s→ν ⟹ λ→τ if it is depicted in the SSR setting. If accepted at their face value, this line of reasoning may appear provocative and so can also be the consequences thereof, including the chain of implications (25). Let us review some practical issues first, before addressing the apparent conceptual controversy, which pops up only under the SSR paradigm, however.

16. CAN INVERSE METER TURNS INTO SECOND AND CYCLE INTO METER?

The eq. (25) and the prototype formula (19) may appear as somewhat controversial from the standpoint of the former unspoken SSR paradigm. As a matter of fact, Mugur-Schächter has already realized that the striking similarity of abstract formal representations of scalar value of linear momentum and of energy in the two elementary functional relationships

$$p = hk = h/\lambda, \;\; E = h\nu = h/T \quad \text{where } k = 1/\lambda \text{ and } \nu = 1/T \tag{27}$$

poses a riddle that should be solved prior to any attempt at grand unification of physics [198]. Besides, functional energy formulas comprising the frequency of light wave do not transform like energy formulas that comprise the frequency of corpuscular wave [198] p. 542. These are good examples of convergence of scientific knowledge [199].

The latter fact hints at the necessity to split the heterogeneous basis into some two dual reciprocal homogeneous bases. Moreover, the unquestionably valid formulas (27) relating spatial and temporal frequencies (namely the reciprocals k,ν) to their respective periods (namely the intervals $\lambda,T = \tau$), could also be perceived as theoretical hints subtly suggesting that the dual reciprocal spaces should be paired with their respective primary spaces. Notice that the relations (27) are not restricted to only particulate matter, for even in electromagnetic theory the momentum manifests itself as radiation pressure [200]. Recall that indeterminacy principle also hints at definitely reciprocal relationship between determination of momentum and position and so it indicates that perhaps position and momentum are not really complete description of the particles involved [201].

De Broglie already wondered how to reconcile the corpuscular structure of light with the fact that in the phenomena of interference and diffraction the luminous energy is distributed as square of the supposed luminous vibration at each point, which can boil down to functional analogy of the momentum and energy, or perhaps even more fundamental analogy between the wavelength λ of the light wave and its period interval $T = \tau$ [202].

If the hypothetical chain of formulas (25) is united with the unquestionably valid formulas (27) in the spirit of the prototype formula (19) then one could propose the following pairings

$$p\rightarrow E \;\; \Longrightarrow \;\; k\rightarrow \nu \;\; \Longrightarrow \;\; \lambda\rightarrow T \tag{28}$$

which again suggest that the same wavicle might be perceived as a single physical entity depicted in some sets of two mutually paired dual reciprocal spaces in two mathematical representations. Yet the essentially 3D magnitudes must also fit 4D quasispatial structures.

Recall that in 4D spacetime the wave can also be expressed in the traditional notation as

$$\exp(\mathbf{k}\bullet\mathbf{r} - \omega t) \;\; \Rightarrow \;\; \mathbf{K} = \left[\mathbf{k}, \frac{\omega}{c}\right] \;\; \Rightarrow \;\; \mathbf{P} = \left[\frac{E}{c^2}\mathbf{c}, \frac{E}{c}\right] = \hbar\mathbf{K} \tag{29}$$

where **P** and **K** are 4-vectors that correspond to the 3-vectors **p** and **k**, respectively, and the **c** denotes velocity of light in its 3-vector form of the scalar speed of light c in vacuum [203]

p. 422. Yet **K** is the [spatial, i.e. lengthlike] frequency 4-vector and **P** is the corresponding to it momentum 4-vector, E = hν = ℏω is the energy of the wave/photon that is expressed in terms of the scalar temporal frequency ν, and the Planck constant $\hbar$ = h/2π is reduced quantum of action [203] p.422. Note that, in order to comply with the angular velocity that is specified as an angle per elapsing time, the wave vector should be scalarly multiplied by the radius vector **r** that points to the wave curve. The other curiosity is the vectorial character of the velocity of light **c**, which makes the vectorial notation of the formulas (29) to appear also operationally consistent, even though the scalar speed of light in vacuum is supposed to be the same in all spatial directions per special relativity. But the latter curiosity clearly corroborates also the LHS of the functional mappings (28).

Notice that the physical action is conventionally interpreted as either energy times time (E⊙T) or as linear momentum times length (p⊙λ) or as angular momentum times angle (L⊙θ) [196] p. 51. The eqs. (27) and the resulting from them relations are nothing new and are indisputable in both classical and quantum mechanics. What seems remarkable, however – as Mugur-Schächter has pointed out – is that the three formal mathematical representations of the same action appear in their own reference frames (spatial and temporal, or lengthlike and timelike, respectively) and that the frames are analogous as if each of them is cast (or immersed) in a separate geometrical setting. Yet the special-relativistic relations (29) suggest that both the lengthlike and timelike setting could be depicted as being 'spatial' in a geometric sense of the latter term.

Thus, both the lengthlike – usually referred to as spatial – and the temporal settings could either be shown as unfurled in a 3D basis of the primary space or as furled in a 1D basis of the secondary space that is paired with the chosen 3D primary space.

17. RELATING 3D ELLIPTIC EQUATIONS TO 4D HYPERBOLIC ONES

Let us consider approach once discussed by Hadamard. Though the 3D Laplace equation

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \tag{30}$$

which is elliptic equation, has unique solution when for points that lie within the given volume the values of the function are given at every point of the boundary surface enclosing the volume, i.e. for Dirichlet problem. But the equation cannot be uniquely solved for the 3D function u(x,y,z) with Cauchy's problem (i.e. when values are given for both the given function u(y,z) = φ(y,z) and for its derivative (∂u/∂x) = ψ(y,z) for the plane x = 0) [204] p. 4f.

Yet the Cauchy's problem is solvable for the 4D hyperbolic equation of spherical waves

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - \frac{\partial^2 u}{\partial t^2} = 0 \tag{31}$$

where the function u(x,y,z,t) now includes also steadily elapsing time parameter t [204] p. 8.

If considered together, the eqs. (30) and (31) provide a subtle theoretical hint that perhaps a single 3D Euclidean space is insufficient for complete description of changing 3D geometric object. Since derivative/differential designates any change, no matter how elaborately evaluated, the change apparently requires an extra running parameter, such as steadily or variably elapsing time. Nevertheless, if the time parameter is supposed to be independently varying as well, then where the elapsing time parameter – if it is varying independently of the variables (x,y,z) constituting the 3D primary space – could actually reside?

Note that the hyperbolic eq. (31) for a wave function P(x,y,z,t) = 0 can also be written as

$$\left(\frac{\partial P}{\partial x}\right)^2 + \left(\frac{\partial P}{\partial x}\right)^2 + \left(\frac{\partial P}{\partial x}\right)^2 - \frac{1}{c^2}\left(\frac{\partial P}{\partial t}\right)^2 = 0 \qquad (32)$$

which is a partial differential equation of the first order [204] p. 10. Here the parameter c is the speed of light in vacuum as usual. Hadamard then wrote that "It must be verified by the function P. When this holds, P(x,y,z,t) = 0 is said to be characteristic of the given equation" and declared that for the eq. (31) such characteristics exist (i.e. they are real) [204] p. 10. Abstract geometric structures and algebraic structures are compared in 4D setting in [205] and in reference to reciprocity relations of spacetime in [206].

Notice that although the term 'real' in mathematics means the opposite to 'imaginary' as the latter term is understood in complex analysis, both terms can refer to realistic objects. It is because the SSR paradigm originally disregarded existence of alternative representations that the term 'imaginary' was coined. Consistency of algebraic operations has virtually compelled some mathematicians to provisionally accept the MSR paradigm, which they at first resisted and thus declared the "undesirable" – yet operationally unavoidable – reality existing beyond the reach of the SSR paradigm as imaginary, even though they could not really regard it as being just as realistic as the 'real' mathematical realm.

The custom of opposing real and imaginary numbers, in which the existence of the latter magnitudes was sometimes considered as being somewhat unreal, ceases to be meritless and can again become meritorious under the MSR paradigm, provided nonexistence of an object in the primary space would automatically imply its "real" existence within the secondary space associated with the primary one.

For the eq. (32) resembles zero-valued classical Lagrangian density of a scalar field P, which in turn satisfies simplified wave equation (v = 1)

$$\frac{\partial^2 P}{\partial t^2} = \nabla^2 P(x,y,z) \qquad (33)$$

that is used in the development of the ontological interpretation of bosonic fields considered in nonrelativistic setting of the De Broglie-Bohm's alternative to the conventional quantum mechanics [105] p. 239.

This venue of inquiry shall be further explored elsewhere for the causal interpretation pursues wave function equipped with both the classical scalar potential and a certain extra quantum potential, the presence of the latter would only unnecessarily complicate matters at this point. In other words: Although I endorse the causal interpretation not because it aims at making quantum mechanics more determinate but rather because it virtually supports multispatiality without ever stating that support explicitly, I shall be focusing in the present paper on formal mathematical deductions.

Moreover, since no wavelength or frequency is present explicitly in traditional approach to geometrical mechanics [207], the purely mathematical alternative to making physics of abstract fields consistent with conceptual mathematics seems to be taking a multispatial approach to representation of basically mathematical objects such as potentials, necessary.

Although neither Hadamard nor anybody else, for that matter, (to the best of my knowledge, of course) ever considered the possibility that elliptical and hyperbolic features could be seen as complementary, then it is conceivable – since we are dealing with spherical waves – that the elliptical and hyperbolic features should be somehow tied together in a prospective expanded mathematical realm. Such an expansion would not be just a certain superposed linear extension, but rather a superimposed umbrella of abstract multispatial hyperspace. It would require adding an extra space associated with the primary space in which the function P(x,y,z,t) could be represented in two abstract homogeneous 3D orthogonal bases rather than in a single heterogeneous 4D basis so that the signatures of these spaces would always be definite and thus quite unambiguous.

Although the use of the letter P to identify both the unfurled primary space and the wave function in the Hadamard's rendition in eq. (32) is accidental, the designation 'P' is indeed too convenient to forfeit it. The need for having an extra space is virtually supported also by the Hilbert's theorem which states that a complete analytic surface, free from singularities, with constant negative Gaussian curvature cannot exist in 3D Euclidean space [208]. Yet they can actually exist in four dimensions, which require either (3+1)D or (1+3)D quasispatial geometric structures, that can be identified with either 4D spacetime or 4D timespace as a counterpart of the former, provided their bases are heterogeneous.

Recapitulation: The conceptual importance of this expose of Hadamard is that usually we tend to avoid nonlinear expressions, which is frequently done in order to maintain the linearity that permits us to superpose solutions of the resulting equations in question. However, a finite superposition of plane waves is insufficient to obtain a localized function if one wants a superposition of uncountably many plane waves with different momenta **k** [196] p. 57. Hence, we may face a different, altered perception of linearity under the prospective MSR paradigm, whose deployment admits paired reciprocal spaces. Linearity could simplify some issues also under the MSR paradigm for it can ensconce unchanged frequency of harmonic vibrations in situations involving time-independent parameters [209].

18. THE LINE ELEMENT IN 4D SPACETIME AND IN OVERLAID 4D TIMESPACE

Let us consider two overlaid 4D quasispatial structures, each containing two primary 3D spaces arranged as paired dual reciprocal spaces representing an abstract yet physically meaningful vacuum field. Since the field's potential energy could neither be created nor annihilated by simply altering some abstract mathematical basis of the space in which the energy is being expressed, changing the operators used to represent the energy within the given geometric structures can affects the structures but not their substance, which is potential energy. If the physics is correct, then this reasoning implies that the two distinct quasispatial 4D structures should be overlaid upon each other.

The overlay is thus asymmetric because the dimensionality of realistic single geometric space cannot exceed 4D per Abel and Galois [33], [36]. As for dimensionality of those abstract sets that are not geometric structures, the restriction is irrelevant, for orthogonality of abstract nD algebraic structures is often immaterial for $n>4$.

By analogy to the prototype formula (18) and since the equality of furled 1D parameters (paired with their unfurled primary 3D variables) could be inferred as being uniquely identifiable indeed, I dare to conjecture the following interspatial overlay prototype schema:

$$interval: \left|\vec{\nabla}^2 [\![\mathrm{R}]\!]\right| \Rightarrow point: \left|\not\nabla^2 [\![\text{ʁ}]\!] \frac{1}{\mathrm{r}}\right| \Leftrightarrow point: |\not\nabla^2| [\![\Lambda]\!] \Leftarrow interval: |\vec{\nabla}^2 [\![\mathcal{T}]\!]| \quad (34)$$

which is the formal equivalence of furling of 3D intervals into the 1D space comprising dual reciprocal points. It corresponds to 3D intervals and unfurling (of the dual reciprocal 1D points into their 3D intervals). The squared prototype schema is shown in terms of paired differential operators: 3D primary (for unfurling) and 1D secondary (for furling), namely $\vec{\nabla}$ or ∇ and $\not\nabla$ respectively.

Although it seems impossible, at least at this time, to determine the formal relationship existing between the 3D unfurled representations that characterize intervals (i.e. line segments in orthogonal straight-linear setting), if the preservation of lengths of intervals (i.e. isometry) is supposed to hold true, the 1D furled representations that involve points (i.e. zero-dimensional geometric entities) appears as holding the key to procedural equivalence of the differential operations: (3D⟹1D) ⟺ (1D⟸3D) shown in (34). Note that our standard of judgment in (34) is admittedly lower because (34) is just a prototype schema, not an exact calculational formula. For the reader which may be confused as to how differential of 0D point can yield 1D result, recall that our point actually represents dual reciprocal variable (i.e. the inverse 1/r of an interval r), in which case this particular operation of differentiation raises the geometric dimensionality in reverse order (i.e. downwards) as in $d(1/r) = -(1/r^2)$ so that it really maps 0D⟶1D when it comes to the object's dimensions or in terms of differential forms. The curious failure of Lagrange resolvents method (mentioned in [33]) is consistent with this peculiar mapping. That failure in conjunction with tacitly suppressed – even though not contested on merit – inconvenient achievements of Abel and Galois prompted me to come up with the idea of pairing of dual reciprocal spaces.

19. REPRESENTATION OF LENGTH-BASED LINE ELEMENT IN 4D SPACETIME

Yet in quest for truth, the convenience of simplifications must not become our overriding objective. Since reciprocity in conjunction with unrestricted division by zero demands at least an operational role for infinity, even in the SSR setting we cannot restrict ourselves to ancient confines of the dogmas that are entrenched in traditional mathematics. Yet because de Broglie developed his idea of wave-particle duality under the SSR paradigm, we should forget for now about possible interspatial ramifications and adopt only the traditional special-relativistic approach, mainly for the sake of making our upcoming comparisons backward compatible.
In geometric vector notation in 4D spacetime (ST), the squared regular line element ds is

$$\boxed{\nabla}^2 \coloneqq \vec{\nabla}^2 + (\mathrm{i}\not\nabla)^2 \quad \Rightarrow \quad (ds)^2 = \left(\vec{X}dx\right)^2 + \left(\vec{Y}dy\right)^2 + \left(\vec{Z}dz\right)^2 + (i\vec{\imath}cdt)^2 \qquad (35)$$

with unit vectors marked by arrows in the 3D reference frame that has, beside the variables x,y,z, of the 3D LBS also the – assumed as steadily elapsing – time parameter t that is rendered as cdt in the 4D ST – compare [ja].

Notice that the imaginary unit i appears twice: once as unit vector $\vec{\imath}$ of the furled time cdt in the 4D formation and again as the scalar imaginary unit î playing the role of operator indicating that the elapsing time parameter t under the furled 1D operator $\nabla\!\!\!\!\mid$ is a foreign element with respect to the native 3D geometric basis $(\vec{X}, \vec{Y}, \vec{Z})$. The variables x,y,z, are subject to the geometric unfurled homogeneous 3D operator $\vec{\nabla}$ while the 4D symbol $\boxed{\nabla}$ denotes the heterogeneous 4D diferential operator in the 4D ST. The essentially special-relativistic eqs. (35) were conceived only for free space, in absence of physical fields. Special-relativistic squared interval is discussed in the SSR setting in [210] p.17, [211].

Since $\left(\vec{X}\right)^2 = \left(\vec{Y}\right)^2 = \left(\vec{Z}\right)^2 = (\vec{\imath})^2 = (\hat{\imath})^2 = i^2 = -1 = |\hat{\imath}\vec{\imath}| = |i\vec{\imath}| \Longrightarrow (i\vec{\imath}cdt)^2 = (cdt)^2$ then the length-based differentials dx,dy,dy, are 3D geometric vectors rather than scalar coefficients. Hence the apparently algebraic 4D signature (+ + + –) actually turns into the effective 4D quasigeometric signature (– – – +) in the 4D heterogeneous quasigeometric basis $(\vec{X}, \vec{Y}, \vec{Z}, i\vec{\imath}c)$ wherein the elapsing time parameter t represents furled foreign 1D quasivector. Here again c is the speed of light in vacuum. Notice that icdt is an imaginary/foreign scalar, but it is supposed to represent a time interval and thus should be expressed by vector instead, for intervals are natural vectors.

This is a minor inconsistency of the mathematics developed under the SSR paradigm – see traditional approach to special theory of relativity [211]. The temporal term $i\vec{\imath}$cdt denotes the imaginary/foreign vector. It is consistent with the multispatial prototype formula (19).

Hence algebraic value (or abstract length) of the squared line element ds in the 4D ST is

$$|\boxed{\nabla}^2| \coloneqq |\vec{\nabla}^2 + (\mathrm{i}\nabla\!\!\!\!\mid)^2| \quad \Rightarrow \quad |(ds)^2| = -\,(dx)^2 - (dy)^2 - (dz)^2 + (cdt)^2 \tag{36}$$

with the effective quasigeometric/algebraic signature (– – – +), just as one would expect.

The d'Alembertian operator is sometimes misdefined as just $\square$ – rather than as squared $\square^2$ as it should be denoted given its squared evaluation – is not really a differential operator but just value of the 4D differential operator $\boxed{\nabla}^2$ that yields the 4D line element given as

$$\square^2\, \mathrm{s} \leftrightarrow |\boxed{\nabla}^2 s| = |(ds)^2| = -\,(dx)^2 - (dy)^2 - (dz)^2 + (cdt)^2 \leftrightarrow \vec{\nabla}^2 l + \nabla\!\!\!\!\mid^2 (ct) \tag{37}$$

with the intent to clarify the confusion of the often confusing traditional notation. Here the symbol ⟷ could be read as "identified with", as opposed to the equivalence ⟺. The dl^2 is identfied with 3D geometric Laplacian $\vec{\nabla}^2$of the scalar value of the length-based interval l:

$$(dl)^2 = -\,(dx)^2 - (dy)^2 - (dz)^2 \leftrightarrow \vec{\nabla}^2 l \tag{38}$$

with the identification shown in an algebraic form, not the conventional geometric notation. Notice that (37) represents an interval not wave for if it would be wavelike equation then its temporal term would have to be dt/c not cdt

The formula (37) can be tentatively rewritten in terms of the differential operators alone

$$\square^2\, \mathrm{s} \leftrightarrow |\boxed{\nabla}^2 s| \leftrightarrow \vec{\nabla}^2 l + \nabla\!\!\!\!\mid^2 (ct). \tag{39}$$

Since intervals are magnitudes of vectors, the conventional notation disregarding the unit vectors (**j**,**k**,**l**) of the local geometric basis cannot be omitted at whim. Hence the minuses are a must because the values of the squared unit vectors yield $\mathbf{j}^2 = \mathbf{k}^2 = \mathbf{l}^2 = -1$. According to the chain of eqs. (36) the eq. (38) is correct. Therefore the 3D the nabla operator ∇ and some other differential operators must be redefined for the MSR setting, which shall be done elsewhere.
The need to treat higher-dimensional hyperspaces upon different foundation is imperative for several reasons, the main of which is because the global existence theorem that was proven for n>5D [212] is not valid in 3D [213].

20. REPRESENTATION OF TEMPORAL LINE ELEMENT IN 4D TIMESPACE

The idea of spatialization of time flow was entertained by many scientists in the past [214] and is still contemplated, mainly by physicists – see [215], [216], for instance. But because of the unspoken yet unquestioned SSR paradigm, the pursuit of the idea remains fruitless. The MSR paradigm with unrestricted division by zero laid the proper mathematical foundations for both structural and operational considerations of the idea. Possibility of quite unrestricted division by zero clears the previous confusion surrounding feasibility of transition between zero and infinity and the acceptance of multispatiality in its extra role as the possibility of dual reciprocal representations of functions as well as geometrized functionals. If treated as operators, the imaginary units i and $\vec{\imath}$, algebraic and geometric, respectively, are independent of bases of the spaces in which they dwell; we may use them in any space. In 4D timespace (TS), as quasigeometric temporal analogue of the spacetime, the squared temporal line element dⲒ (i.e. differential of a temporal vectorlike interval Ⲓ) can thus be written as

$$(d\text{Ⲓ})^2 = \left(\vec{F}df\right)^2 + \left(\vec{G}dg\right)^2 + \left(\vec{H}dh\right)^2 + \left(i\vec{\imath}(\tfrac{1}{c})d\Lambda\right)^2 \tag{40}$$

which is analogous to eq. (35). Eq. (40) depicts 4D differential of Ⲓ ("based T") shown in the orthogonal 3D homogeneous basis $\vec{F}, \vec{G}, \vec{H}$, that is made heterogeneous by addition of the, foreign to ($\vec{F}, \vec{G}, \vec{H}$,) imaginary operator $\vec{\imath}$. The $(d\text{Ⲓ})^2$ is squared 4D quasitemporal line element with 3D temporal unit vectors $\left(\vec{F}\right)^2 = \left(\vec{G}\right)^2 = \left(\vec{H}\right)^2 = -1$. The scalar temporal differentials df,dg,dh, represent spreading out temporal intervals in the homogeneous basis (f,g,h) native to the 3D TBS within the 4D TS. Notice that the parameter Λ depicted above is foreign with respect to the native basis (f,g,h) of the 3D TBS just as the elapsing time parameter t is foreign to the length-based native basis (x,y,z) of the 3D LBS immersed in the 4D ST. The value (or length/norm) of the squared temporal interval dⲒ (i.e. line element) in the 4D TS is given as

$$\square^2\,\text{Ⲓ} \leftrightarrow |\boxed{\nabla}^2\text{Ⲓ}| = |(d\text{Ⲓ})^2| = -(df)^2 - (dg)^2 - (dh)^2 + \left(\tfrac{1}{c}d\Lambda\right)^2 \leftrightarrow \vec{\nabla}^2\mathcal{T} + \not\nabla^2\frac{\Lambda}{c} \tag{41}$$

with the effective quasigeometric/algebraic signature (– – – +), just as in (36). The squared homogeneous temporal line element d$\mathcal{T}$ of the 3D TBS is temporal analogue of the eq. (38)

$$(d\mathcal{T})^2 = -(df)^2 - (dg)^2 - (dh)^2 \leftrightarrow \vec{\nabla}^2\mathcal{T} \tag{42}$$

which is temporal counterpart of the usual geometric (or "spatial") 3D Laplacian (38). Thus, consistency of operations demands multispatial representations of objects depicted in heterogeneous bases without making existential postulates. Notice that the derivations were deduced from de Broglie/Einstein's functionals.

The formula (41) can be tentatively rewritten in terms of the differential operators alone

$$\square^2\, \mathbb{I} \leftrightarrow |\boxed{\nabla}^2 \mathbb{I}| \leftrightarrow \vec{\nabla}^2 \mathcal{T} + \nabla\!\!\!\!/^{\,2}\left(\frac{A}{c}\right) \tag{43}$$

where the temporal line elements (40), (41), (42), (43) resemble wavelike equation.

By comparing the above Darboux matrix to the eq. (42) in conjunction with (40) one can conjecture that the intervals f,g,h, of the 3D TBS that is immersed in 4D TS are functionally depending on scalar magnitudes v,κ,τ, respectively. The latter scalars belong in the usual 3D LBS that is immersed in 4D ST. If the scalar velocity/speed is denoted by v = |**v**|, and κ is the principal normal curvature and τ is torsion (or binormal curvature), then according to the differential-geometric naming convention, the conjectured here function-based additive dependence and its functional dependence (signified by square brackets) could be written as

$$\left(d\mathcal{T}([\mathrm{v},\kappa,\tau])\right)^2 \coloneqq -\,(\mathrm{df}[\mathrm{v}])^2 - (\mathrm{dg}[\kappa])^2 - (\mathrm{dh}[\tau])^2 \leftrightarrow \vec{\nabla}^2 \mathcal{T}[\mathrm{v},\kappa,\tau] \tag{44}$$

which is like function-based dependence spreading in three temporal directions: tangential f (1st derivative v), normal g (2nd derivative κ), and binormal h (3rd derivative τ) as it is concisely explained above in section 13 and in [186]. Recall that smooth curve with κ≠0 is completely determined by its curvature and torsion up to position [217]. To quickly grasp the meaning of the formula (41) let us consider simpler Darboux vector **w** = τ**T** + κ**B** so that we can also write

$$\dot{\boldsymbol{T}} = \boldsymbol{w} \times \boldsymbol{T} \tag{45}$$

$$\dot{\boldsymbol{N}} = \boldsymbol{w} \times \boldsymbol{N} \tag{46}$$

$$\dot{\boldsymbol{B}} = \boldsymbol{w} \times \boldsymbol{B} \tag{47}$$

with dotted derivatives (with respect to either time *t* or arclength *s*) of the unit vectors **T**,**N**,**B**, in a trihedron moving at unit speed along the path with Darboux vector **w** [218]. Notice that function-based dependence could be more complicated (as Frenet and Darboux matrices indicate) especially for nonunit constant or for varying speed trajectory path.

Just as the arclength function s() represents spatial trajectory curve l in 3D LBS, the function $\mathbb{I}$() represents 4D temporal trajectory curve (depicted in terms of temporal interval $\mathcal{T}$ in 3D TBS) that corresponds to the spatial trajectory curve l. The functional's (understood here as noun, not adjective) dependence is marked by square brackets to distinguish it from function-based dependence, which it does not preclude, but to claim also function-based dependence I would have to show that the latter is plausible indeed. This shall be done elsewhere as it requires us to redefine some algebraic and differential operators.

Although the moving Frenet/Darboux frames describe kinematics of the path, they ignore the dynamics of the given motion. Motion of hydrodynamical flow was discussed in [219] and in quaternionic setting in [220].

Instead of scalar curvature κ one can define also curvature vector $\boldsymbol{K}(t) = D_s\,\boldsymbol{T}(t)$ where **T** is the unit tangent vector, s is the arclength along the given path and D_s denotes the directional derivartive along the path. By the chain rule we can write: $D_t\,\boldsymbol{T}(t) = [D_s\,\boldsymbol{T}(t)]D_t s$ and then replacing $D_t s = D_t R(t)$ we obtain $\boldsymbol{K}(t) = \frac{D_t \boldsymbol{T}(t)}{|D_t R(t)|} = D_s\,\boldsymbol{T}(t)$ where R(t) is the radius pointing to the path – compare [221]. Recall that directional derivative is the rate of change of the function Φ that describes the trajectory path with respect to the arclength along the path so that: $\frac{d\Phi}{\mathrm{ds}} = \nabla\Phi \bullet \frac{\mathrm{d}\mathbf{R}}{\mathrm{ds}} = \nabla\Phi \bullet \mathbf{T}$, compare [222].

Hence the essentially abstract mathematical conjecture (44) obviously has undeniable direct physical significance too. If so, then perhaps de Broglie's wave-particle duality actually can imply much more than even he himself ever realized.

21. INTEGRATED LINE ELEMENTS YIELD GEOMETRIZED FUNCTIONALS

For the physically relevant conjecture (44) is just the 3D component of the 4D relation (41), which (on integration) evidently yields the equivalent of special-relativistic proper time

$$\text{Ɪ} = \int\sqrt{|\boxed{\nabla}^2 \text{Ɪ}|} = \int\sqrt{|(d\text{Ɪ})^2|} = \int\sqrt{[-(df)^2 - (dg)^2 - (dh)^2 + \left(\frac{1}{c}d\Lambda\right)^2]} \qquad (48)$$

as temporal lapse of proper time Ɪ expressed in a primary temporal reference frame though. The conjecture (48) can also be abbreviated and rewritten in terms of differential operators as

$$\text{Ɪ} = \int\sqrt{|\boxed{\nabla}^2 \text{Ɪ}|} \leftrightarrow \int\sqrt{(\vec{\nabla}^2 \mathcal{T} + \forall^2 \frac{\Lambda}{c})} \qquad (49)$$

even though the latter prototype formula is tentative because the nabla operator is imprecise.

The usual special-relativistic proper time is given by similar integral with 4D integrand

$$ɟ = \int\sqrt{|\boxed{\nabla}^2 s|} = \int\sqrt{|(ds)^2|} = \int\sqrt{[-(dx)^2 - (dy)^2 - (dz)^2 + (cdt)^2]} \qquad (50)$$

that yields the lapse of the proper time ɟ in primary time-based "spatial" reference frame though, compare [223].

Since the letter τ 'tau' is already used to denote torsion, I will use the inverted ɟ to symbolize proper time, because τ is set in the spatial reference frame (x,y,z,ct) whose spatial part is foreign (hence virtually inverted) with respect to the native temporal reference frame (f,g,h,Λ/c), which would normally be used for calculating temporal variables, were it not for the fact that neither former mathematics nor physics ever recognied the need to maintain conceptual consistency and to adopt the MSR paradigm.

The conjectured formula (50) also can be abbreviated and rewritten in terms of the two paired differential operators as

$$ɟ = \int\sqrt{|\boxed{\nabla}^2 s|} = \int\sqrt{(\vec{\nabla}^2 l + \forall^2 (ct))} \qquad (51)$$

even though the latter prototype formula is tentative due to the fact that both the 3D nabla operators ∇ and $\vec{\nabla}^2$ are still too imprecisely defined even for the traditional SSR paradigm.

From the formulas (41) and (44) in conjunction with the formula (48) I can conjecture also

$$\mathcal{T} = \int\sqrt{\left(d\mathcal{T}([\mathrm{v},\kappa,\tau])\right)^2} = \int\sqrt{(-\,(\mathrm{df}[\mathrm{v}])^2 - (\mathrm{dg}[\kappa])^2 - (\mathrm{dh}[\tau])^2)} \leftrightarrow \int\sqrt{(\vec{\nabla}^2\mathcal{T}[\mathrm{v},\kappa,\tau])} \quad (52)$$

$$\mathrm{I} = \int\sqrt{|\boxed{\nabla}^2\mathrm{I}|} = \int\sqrt{|(d\mathrm{I})^2|} = \int\sqrt{[-(\mathrm{df}[\mathrm{v}])^2 - (\mathrm{dg}[\kappa])^2 - (\mathrm{dh}[\tau])^2 + \left(\frac{1}{c}d\Lambda\right)^2]} \quad (53)$$

and the latter formula can also be rewritten in terms of the differential operators as follows

$$\mathrm{I} = \int\sqrt{|\boxed{\nabla}^2\mathrm{I}|} \leftrightarrow \int\sqrt{(\vec{\nabla}^2\mathcal{T} + \not{\nabla}^2\frac{\Lambda}{c})} \quad (54)$$

with dependence on the functionals that determine the curve [217]: speed, curvature and torsion. The speed, curvature and torsion v,κ,τ, are separated in 3D LBS because the native homogeneous 3D basis of the LBS is orthogonal. The reason for having them separated also in the TBS is that we use orthogonal homogeneous native temporal basis also in the TBS. Notice that even the (temporal) Λ, which is virtually treated as a arclengthlike parameter in the 4D TS, should actually also be codetermined by the speed/velocity, curvature and torsion v,κ,τ, that characterize the trajectory path, if it were depicted directly within the 3D LBS that is immersed within the 4D ST. Although this supposition seems plausible, the dependence on v,κ,τ, is nonetheless just a conjecture yet.

The conjectured above formulas and schemas conform to the – deduced via syntheses of operators – special-relativistic representations (37) and (41) in compliance with the 4D formula (35) determining the multispatial 4D differential operator $\boxed{\nabla}$. The conjectured here prototype formulas (52), (53), (54), also hint at the need to expand the STR and suggest that the general theory of relativity (GTR) can be extended to include artificial accelerations too.

This investigation cannot continue beyond this point, however, because intervals can be treated as either scalars or vectors. Although some authors proposed different symbol for vector Lplacian [224], [225], this and a few related issues need more than just assignment of a different symbol. Most differential operators for deployment in a multispatial setting require more finegrained definitions, which shall be accomplished elsewhere.

22. SOME COMMENTS ON DUAL RECIPROCAL REPRESENTATIONS

Although some authors believe the dogma that accelerations do not affect the time rate accumulated by moving clocks [210] p. 27, [226], their conclusion is based on short lasting experiments. If gravitational and artificial acceleartions can change velocity then the change of resulting time rates could be attributed to the applied accelerations. Reconciliation of long running experiments with atomic clocks flying around the world showed that the theoretical dogma is untenable [227]. Moreover, since airplanes tend to fly in practically the same gravitational potential (once they ascended to the designated altitude of their cruising path), and after that fly at practically constant speed relative to the earth's surface, then their accumulation of still varying time rates must have been influenced by accelerations induced by nonradial effects attributed to the earth's density [66], [82], [83], [239-241]. Moving trihedrons of Frenet or Darboux are not restricted to spatial trajectories alone but could be applied to temporal trajectories too. Since curvature and torsion determine curvature of trajectories/paths (and thus also bending of the space in which these paths are immersed) – and the elapsing time as well, when relativistic intervals are evaluated – then the aforesaid dogma is not conceptually binding. For if something is true, it should be possible to calculate it in several ways, not just according to the dogma, which could not even approximately explain the cause of the East-West asymmetry dutifully recorded by the flying clocks [227].

Since the earth spins and the airplanes were flown in two opposite directions, their kinematics was more diversified than the dynamics of the airplanes carrying the clocks. By the way, the discrepancy known as the East-West asymmetry hinted at yet another – experimentally discovered – nonradial effect, which also remained unexplained due to the dogma. The extra nonradial effect of gravity reconciled also other formerly unreconciled and unanticipated experimental results [150], which are explained also for nonspecialists in [83].

While the regular Einstein's time retardation factor γ depends only on constant speed v

$$\gamma = \sqrt{(1 - \frac{v^2}{c^2})} \tag{55}$$

with v = |**v**| = const [228], [229], which is also called the Lorentz factor of STR, the derived above formulas suggest the possibility of expansion just as did some experiments [83], [227].

The fact that the prototype formulas (52), (53), (54) promise to expand the special-relativistic time retardation factor onto constant accelerations is not unimaginable because the formulas depend also on curvature and torsion, i.e. on the second and third derivatives of the radius vector **r** pointing to the inverse, dual reciprocal temporal trajectory. Their being of an inverse nature, however, was theoretically quite unanticipated, at least not before Newton-Wigner wavefunction has been formulated [230], I guess. For, at first glance, their inverse character might appear as contradicting the regular special-relativistic time retardation factor. Besides, the Newton-Wigner wavefunction reveals nonlocal properties [231-233], which are commonly considered as strange, and is also noncovariant [234].

Macroscopic systems show usual time dilation/retardation in motion such that clocks moving at constant speed accumulate time slower by the Lorentz factor γ according to the eq. (55). However, with the help of Newton-Wigner wavefunction Mikhail I. Shirokov showed that the usual special-relativistic explanation of the Einstein retardation based on Lorentzian transformations of position and time is inapplicable to quantum wave packets considered as relativistic clocks [235]. Although his conclusion was discounted for lacking causality due to its use of the nonlocal (i.e. allegedly unphysical) Newton-Wigner wavefunction [236], nonlocality is natural feature under the MSR paradigm. Hence the possibility of expansion of the traditional physics implied by the prototype formulas derived above is not uncorroborated even though at first glance it may appear controversial.

23. CONCLUSIONS

Analyzing Green's function involving the quantum-mechanical wave function Ψ and its conjugate, de Broglie concluded that his idea of wave-particle duality came to fatal conflict with his conception of the physical wave u having singularity that represents the particle of the wave-particle duo, for the wave function was equal to zero at the point where the particle should be found according to probabilistic interpretation of the wave function. However, if the wave function and its conjugate would be housed in two separate yet paired dual reciprocal spaces, then zero in the primary space would correspond to infinity hosted in the dual reciprocal space, which is paired with the primary space provided that we accept the reciprocal relationship between zero and infinity: $\frac{1}{0} = \infty$.

Therefore, finding the particle would be ensured because the probability of finding the particle would be equal to one (once it is normalized), provided the imaginary unit would be understood as an interspatial operator. Hence the wave-particle duality and its consequences make perfect sense once the de Broglie ideas are cast within the new synthetic mathematical framework of multispatial structures instead of being considered solely under the traditional single-space reality paradigm that de Broglie and most of his contemporaries took for granted.

The wave-particle duality has been recast mathematically as 6D wavicle depicted in two paired 3D dual reciprocal spaces. The pairing is implemented via immersing the 3D spaces in two 4D quasispatial structures resembling (3+1)D = 4D spacetime overlaid by (1+3)D = 4D timespace. Algebraically, the 4D spacetime and the partly overlapping it 4D timespace can form (4+4)D operational quasispatial structure. Whether the conjectured (4+4)D quasispatial structure can be implemented in 8D algebraic set of octonions, and to what degree, still remains an open question at this point.

Although some of special-relativistic functionals suggest the need to expand the special relativity as well, this does not defy the special nor the general relativity, even though both these theories could be expanded in the multispatial setting.

The experimentally confirmed de Broglie's idea of wave-particle duality complies not only with heretofore developed physics but also hints at the possibility of its expansion. It is fair to say that due to conceptual deficiencies of traditional mathematics in general with its insufficiently developed concept of (operational as well as structural) infinity and especially the unwarranted prohibition on division by zero that the former wave mechanics was doomed. The wave-particle duality of 6D = (2×3)D wavicle represented in two paired 4D dual reciprocal quasispaces of a 8D = (2×4)D heterogeneous quasispatial structure requires an abstract multispatial approach in order to be implemented in 4-vectors or 4D quaternions.

References

[1] Poincaré H. L'avenir des mathématiques. FB Editions, 1908, p. 8f.

[2] Poincaré H. Des fondements de la géométrie. Forgotten Books, 2015, pp. 11f, 20, 60.

[3] Hilbert D. Natur und mathematisches Erkennen. Basel: Birkhäuser, 1992, p. 17.

[4] Bellone E. A world on paper. Studies on the Second Scientific Revolution. Cambridge, MA: The MIT Press, 1980, p. 29f.

[5] Gonseth F. Les entretiens de Zurich sur les fondements et la méthode des sciences mathématiques. Zurich, 1941, p. 59.

[6] Krat V.A. (Ed.) Heuristic role of mathematics in physics and cosmology. Leningrad, 1975, [in Russian], p. 31.

[7] Hall R.J. A philosophy of geometry. *Philos. Math*. 2 (1965) 13-31, see p. 23.

[8] Hamilton A.G. Logic for mathematicians. Cambridge: Cambridge Univ. Press, 2000, p. 57.

[9] Fang J. Mathematical existence. *Philos. Math*. 11 (1974) 5-8

[10] Corry L. Modern algebra and the rise of mathematical structures. Basel: Birkhäuser, 1996, p. 391.

[11] MacLane S. Mathematical models of space. *Am. Math. Month*. 68 (1980) 184-191, see p. 191.

[12] [no author was given] Grundlagen der Geometrie und algebraische Methoden. *Potsdamer Forschungen, Reihe B, Heft* 3, 1974, see p. 39.

[13] Šanin N.A. Constructive real numbers and constructive function spaces. Providence, RI: AMS, 1968, pp. 305f, 307.

[14] Šurygin V.A. Constructive sets with equality and their mappings. [in: Orevkov V.P. & Šanin N.A. (Eds.) Problems in the constructive mathematics V. *Proc. Steklov Inst. Math*. 113 (1970) 195].

[15] Thom R. To predict is not to explain. Thombooks Press, 2015, p. 85.

[16] Shlakhin G.G. Mathematics and objective reality. Rostov: Rostov Univ. Press, 1977 [in Russian], p. 115.

[17] Kedrovskii O.I. Methodological problems of development of mathematical knowledge. Kiev, 1977 [in Russian], p. 27.

[18] Thom R. in: Tsatsanis S.P. (Ed.). To predict is not to explain. Thombooks Press: 2015, p. 144.

[19] Vyal'tsev A.N. Discrete spacetime. Moscow, 1965 [in Russian], p. 131.

[20] Panov M.I. Methodological problems of the development and application of mathematics. Moscow, 1985 [in Russian], p. 25.

[21] Gott V.S. Philosophical problems of contemporary physics. Moscow, 1972 [in Russian], p. 13f.

[22] Kaplansky I. Set theory and metric spaces. Boston: Allyn and Bacon, 1972, p. 123f.

[23] McLeod R.J.Y. & Baart M.L. Geometry and interpolation of curves and surfaces. Cambridge: Cambridge Univ. Press, 1998, p. 129ff.

[24] Wallach N.R. Harmonic analysis on homogeneous spaces. Mineola, NY: Dover, 2018, p. 22ff.

[25] Thom R. A panorama of mathematics (1989). [pp. 101-115 in: Tsatsanis S.P. (Ed.). René Thom creator of catastrophe ('tipping point') theory: In praise of mathematics. Thombooks Press, 2017, see pp. 101,115].

[26] Grätzer G. Universal algebra. Princeton, NJ: Van Nostrand, 1968, p. 270ff.

[27] Björk J.-E. Rings of differential operators. Amsterdam: North-Holland, 1979, p. 92ff.

[28] McQuistan R.B. Scalar and vector fields: A physical interpretation. New York: Wiley. 1965, p. 198.

[29] Tai C.-T. Generalized vector and dyadic analysis. Applied mathematics in field theory. Piscataway, NJ: IEEE Press, 1997, pp. 65,73ff.

[30] Penrose R. The road to reality. A complete guide to the laws of the universe. New York: Alfred A. Knopf, 2005, pp. 189,503.

[31] Crowe M.J. A history of vector analysis. The evolution of the idea of a vectorial system. Notre Dame: Univ. of Notre Dame, 1967, p. 186f.

[32] Hausdorff F. Set theory. New York: Chelsea Publishing, 1957, p. 332ff.

[33] Czajko J. Operational constraints on dimension of space imply both spacetime and timespace. *Int. Lett. Chem. Phys. Astron.* 36 (2014) 220-235

[34] Uspensky J.V. Theory of equations. New York: McGraw-Hill, 1948, p. 82ff.

[35] Shurman J. Geometry of the quintic. New York: Wiley, 1997, p. 91ff.

[36] Czajko J. Operational restrictions on morphing of quasi-geometric 4D physical spaces. *Int. Lett. Chem. Phys. Astron.* 41 (2015) 45-72

[37] Olver P.J. Classical invariant theory. Cambridge; Cambridge Univ. Press, 1999, p. 234ff.

[38] Khovanskii A. Topological Galois theory. Solvability and unsolvability of equations in finite terms. Berlin: Springer, 2014, p. 144ff.

[39] Browne J. Grassmann algebra I: Foundations. Exploring extended vector algebra with *Mathematica*. Eltham, VIC, Australia: Barnard Publishing, 2012, pp. 7, 15, 13ff, 19, 39, 49ff, 93ff, 125ff, 161, 217.

[40] Morris D. The naked spinor. A rewrite of Clifford algebra. Port Mulgrave: Abane & Right, 2015, pp. 83,87,175.

[41] Chevalley C. Fundamental concepts of algebra. New York: Academic Press, 1956, p. 125ff.

[42] Deschamps G.A. & Ziolkowski R.W. Comparison of Clifford and Grassmann algebras in applications to electromagnetics. [pp. 501-515 in: Chisholm J.S.R. & Common A.K. (Eds.) Clifford algebras and their applications in mathematical physics. Dordrecht: Reidel, 1986].

[43] MacCamy R.C. & Mizel V.J. Linear analysis and differential equations. London: Macmillan, 1969, p. 195ff.

[44] Czajko J. Quaternionic division by zero is implemented as multiplication by infinity in 4D hyperspace. *World Scientific News* 94(2) (2018) 190-216

[45] Czajko J. Algebraic division by zero implemented as quasigeometric multiplication by infinity in real and complex multispatial hyperspaces. *World Scientific News* 92(2) (2018) 171-197

[46] Czajko J. On Cantorian Spacetime over Number Systems with Division by Zero. *Chaos Solit. Fract.* 21 (2004) 261-271

[47] Pedoe D. Geometry. A Comprehensive Course. New York: Dover, 1988, pp. 244,276.

[48] Bumcrot R.J. Modern Projective Geometry. New York: Holt, Rinehart and Winston, 1969, p. 75.

[49] Verdina J. Projective Geometry and Point Transformations. Boston: Allyn & Bacon, 1971, p. 24ff.

[50] Hopkins E.J. & Hails E.J. An Introduction to Plane Projective Geometry. Oxford: At The Clarendon Press, 1953, p. 5.

[51] Baer R. Linear algebra and projective geometry. New York: Academic Press, 1952, p. 96.

[52] Seidenberg A. Lectures on projective geometry. Princeton, NJ: D. Van Nostrand, 1963, p. 43ff.

[53] Blaschke W. Ein Beitrag zur Liniengeometrie. *Rend. Circ. Mat. Palermo* 33 (1912) 247-253

[54] Blaschke W. Über topologische Fragen der Differentialgeometrie. *Jhb. Deut. Math.-Ver*. 37 (1928) 193-206.

[55] Brauer R. A characterization of null systems in projective space. [pp. 645-52 in: Fong P. & Wong W.J. (Eds.) Richard Brauer – Collected papers III. Cambridge, MA: The MIT Press, 1980].

[56] Levy H. Projective and related geometries. New York: Macmillan, 1964, p. 108.

[57] Griffiths P. & Harris J. Principles of algebraic geometry. New Delhi: Wiley, 2014, p. 705ff

[58] Álvarez-Gaumé L. & Zamora F. Duality in quantum field theory (and string theory). [pp. 46-83 in: Panvini R.S. & Weiler T.J. (Eds.) Fundamental particles and interactions. Woodbury, NY: AIP, 1998, see p. 48ff].

[59] Saaty T.L. & Bram J. Nonlinear mathematics. New York: Dover, 1964, p. 177.

[60] Norio Konno, Localization of an inhomogeneous discrete-time quantum walk on the line, *Quantum Information Processing*, v. 9 n. 3, p. 405-418, June 2010

[61] Lang S. Complex analysis. New York: Springer, 2013, pp. 5,11,76,233.

[62] Bouhennache R. Truly hypercomplex numbers: Unification of numbers and vectors. arXiv, 2014, p. 15.

[63] Snapper E. The three crises in mathematics: Logicism, Intuitionism, and Formalism. [pp. 183-193 in: Campbell D.M. & Higgins J.C. (Eds.) Mathematics. People – Problems – Results. II. Belmont, CA: Wadsworth, 1984, see p. 192].

[64] Irina V.R. & Novikov A.A. In the world of scientific intuition. Intuition and mind. Moscow, 1978, p. 125 [in Russian].

[65] Weber A. & Lide D.R., Jr. Molecular spectroscopy. [pp. 1522-1600 in: Lerner R.G. & Trigg G.L. (Eds.) Encyclopedia of physics: Vol. 2: M-Z. Third, completely revised and enlarged edition. Weinheim: Wiley-VCH Verlag, 2005, see pp. 1523,1532].

[66] Czajko J. Complex Conjugate Time in Macro Experiments. *Chaos Solit. Fract.* 13 (2002) 17-23

[67] Phillips H.B. Vector analysis. New York: Wiley, 1960, p. 82.

[68] Schwartz M., Green S. & Rutledge W.A. Vector analysis with applications to geometry and physics. New York: Harper & Brothers Publishers, 1960, pp. 204,365.

[69] Manes E.G. Algebraic theories. New York: Springer, 1976, p. 139.

[70] Anderson J. & Austin K. Paradigms of proof. *Math. Gaz.* 79 (1996) 489-495

[71] Hintikka J. & Remes U. The method of analysis. Dordrecht: D. Reidel, 1974, p. 119.

[72] Poincaré H. Science and hypothesis. New York: Dover, 1952, p. 50.

[73] Greenberg M.J. Euclidean and non-Euclidean geometries. Development and history. New York: Freeman, 1993, p. 293.

[74] Mooij J.J.A. La philosophie géométrique de Henri Poincaré. *Synthese* 16 (1966) 53

[75] Proietti M. et al. Experimental rejection of observer-independence in the quantum world. arXiv, 2019, see p. 2

[76] Brukner Č. On the quantum measurement. arXiv, 2015.

[77] Brukner Č. A no-go theorem for observer-independent facts. arXiv, 2018.

[78] Cramer J. & Schütte-Nütgen C. Experimental violations of Bell's inequalities. ETHZ, Zurich, 2010, Qudev.

[79] Helms L.L. Introduction to potential theory. New York: Wiley, 1969, p. 237f.

[80] Myrvold W., Genovese M. & Shimony A. Bell's theorem. Stanford, Plato, 2019.

[81] Czajko J. Path-independence of work done theorem is invalid in center-bound force fields. *Stud. Math. Sci.* 7(2) (2013) 25-39

[82] Czajko J. Equipotential energy exchange depends on density of matter. *Stud. Math. Sci.* 7(2) (2013) 40-54

[83] Czajko J. Galilei was wrong: Angular nonradial effects of radial gravity depend on density of matter. *Int. Lett. Chem. Phys. Astron.* 30 (2014) 89-105

[84] Czajko J. Equalized mass can explain the dark energy or missing mass problem as higher density of matter in stars amplifies their attraction. *World Scientific News* 80 (2017) 207-238

[85] Evans M. & Vigier J.-P. The enigmatic photon 2: Non-abelian electrodynamics. Dordrecht: Kluwer, 1995, pp. 44ff,68ff,107ff,123ff,130ff.

[86] Evans M. & Vigier J.-P. The enigmatic photon 1: The field $\mathbf{B}^{(3)}$. Dordrecht: Kluwer, 1994, pp. 5ff,13,35,38,117ff.

[87] Evans M., Vigier J.-P., Roy S. & Jeffers S. The enigmatic photon 3: Theory and practice of the $\mathbf{B}^{(3)}$ field. Dordrecht: Kluwer, 1996, p. 119ff.

[88] Van Hove M.A., O'Keefe M.A. & Marton L. Electron diffraction. [pp. 616-621 in: Lerner R.G. & Trigg G.L. (Eds.) Encyclopedia of physics: Vol. 1: A-L. Third, completely revised and enlarged edition. Weinheim: Wiley-VCH Verlag, 2005].

[89] Andrade e Silva J. & Lochak G. Quanta. New York: McGraw-Hill, 1969, p.118.

[90] De Broglie L. The current interpretation of wave mechanics. A critical study. Amsterdam: Elsevier, 1964, p. 7.

[91] Lochak G. The evolution of the ideas of Louis de Broglie on the interpretation of wave mechanics. [pp. 11-33 in: Barut A.O., Van der Merwe A. & Vigier J.-P. (Eds.) Quantum, space and time – the quest continues. Cambridge: Cambridge Univ. Press, 1983, see pp. 18f,24].

[92] French A.P. Special relativity. New York: W.W. Norton, 1968, p.17.

[93] Corinaldesi E. & Strocchi F. Relativistic wave mechanics. Amsterdam: North-Holland, 1963, pp. 5ff,296f.

[94] Weisstein, Eric W. Klein-Gordon Equation. From *MathWorld*--A Wolfram Web Resource. http://mathworld.wolfram.com/Klein-GordonEquation.html

[95] Katti A.N. The mathematical theory oof special and general relativity. North Charleston, SC: CreateSpace Independent Publishing, 2013, pp. 35,54.

[96] Lanciani P. A model of the electron in a 6-dimensional spacetime. *Found. Phys*. 29(2) (1999) 251-265

[97] Evans M.W. The Evans-Vigier field $\mathbf{B}^{(3)}$ interpreted as a De Broglie pilot field. [pp. 241-254 in: Evans M.W. The enigmatic photon 5: O(3) electrodynamics. Dordrecht: Kluwer, 1999, see pp. 241,243].

[98] Recami E. & Evans M. Physical meaning of the photon's wavefunction according to Ettore Majorana. [pp. 137-145 in: Evans M., Vigier J.-P., Roy S. & Hunter G. (Eds.) The enigmatic photon 4: New directions. Dordrecht: Kluwer, 1998, see p. 144].

[99] Evans M.W. The derivation of the Majorana form of Maxwell's equations. [pp. 235-240 in: Evans M.W. The enigmatic photon 5: O(3) electrodynamics. Dordrecht: Kluwer, 1999, see pp. 238f].

[100] Holland P.R. The quantum theory of motion. An account of the de Broglie-Bohm causal interpretation of quantum mechanics. Cambridge: Cambridge Univ. Press, 1997, pp. 25, 26,89f.

[101] Barut A.O. The fallacy of the arguments against local realism in quantum phenomena. [pp. 9-18 in: Van der Merwe A. & Garuccio A. (Eds.) Waves and particles of light and matter. New York: Plenum Press, 1994, see p. 13f]

[102] De Broglie L. L'Oeuvre d'Einstein et la dualité des ondes et des corpuscules. *Rev. Mod. Phys*. 21(3) (1949) 345-347

[103] Kroemer H. Quantum mechanics for engineers, materials science and applied physics. Englewood Cliffs, NJ: Prentice-Hall, 1994, see Chapter 1.

[104] French A.P. Vibrations and waves. New Delhi: CBS Publishers, 2003, p. 209f,214.

[105] Bohm D. & Hiley B.J. The undivided universe. An ontological interpretation of quantum theory. London: Routledge, 1993, pp. 110,239.

[106] Do Carmo M.P. Differential geometry of curves and surfaces. Mineola, NY: Dover, 2016, pp. 451ff,331ff.

[107] Jones G.A. & Singerman D. Complex functions. An algebraic and geometric viewpoint. Cambridge: Cambridge Univ. Press, 1987, pp. 1ff,4ff.

[108] Paliouras J.D. & Meadows DS. Complex variables for scientists and engineers. Mineola, NY: Dover, 2014, p. 30f.

[109] Kumar D. Advanced calculus of several variables. New Delhi: Narosa Publishing, 2014, p.18.

[110] Bohm D. Wholeness and the implicate order. London: Routledge, 2002, pp. 109f,123ff.

[111] De Broglie L. Non-linear wave mechanics. A causal interpretation. Amsterdam: Elsevier, 1960, pp. 22ff, 24,89ff,99ff,209ff,216ff,220ff,263ff,273,291

[112] D'Espagnat B. Matière et réalité. [pp. 211-225 in: Collongues R. et al. La matière aujourd'hui. Paris: Éditions du Seuil, 1981, see p. 212].

[113] De Broglie L. An introduction to the study of wave mechanics. London: Methuen, 1930, p. 6f.

[114] De Broglie L. Matter and light. The new physics. Dover Publications; 1st English edition (January 1, 1939), p. 60ff.

[115] Gouesbet G. Hidden worlds in quantum physics. Mineola, NY: Dover, 2013, p. 141ff.

[116] Lochak G. De Broglie's initial conception of de Broglie waves. [pp. 1-25 in: Diner S., Fargue D., Lochak G. & Selleri F. (Eds.) The wave-particle dualism. A tribute to Louis de Broglie on his 90th birthday. Dordrecht: Reidel, 1983, see p. 5].

[117] De Broglie L. La physique nouvelle et les quanta. Flammarion, 1992, p. 11.

[118] [118] De Broglie L. Introduction to the Vigier theory of elementary particles. Amsterdam: Elsevier, 1963, pp. 133,137.

[119] Czajko J. Mathematical gateway to complementary hidden variables in macrophysics. *Int. Lett. Chem. Phys. Astron.* 50 (2015) 117-142

[120] D'Espagnat B. On physics and philosophy. Princeton, NJ: Princeton Univ. Press, 2006, p. 123f.

[121] Andersson M. Topics in complex analysis. New York: Springer, 1997, p. 8.

[122] O'Neil P.V. Advanced engineering mathematics. Pacific Grove, CA: Thompson, 2003, p. 1092ff.

[123] Watson G N. Complex integration & Cauchy's theorem. Mineola, NY: Dover, 2012, pp. 31,46.

[124] Page A. Mathematical analysis and techniques II. Oxford: Oxford Univ. Press, 1974, p. 226.

[125] Weisstein, Eric W. Cauchy Integral Formula. From *MathWorld*--A Wolfram Web Resource. http://mathworld.wolfram.com/CauchyIntegralFormula.html

[126] Okubo S. Introduction to octonion and other non-associative algebras in physics. Cambridge: Cambridge Univ. Press, 1995, pp. 2,11ff.

[127] Barr T.H. Vector calculus. Upper Saddle River, NJ: Prentice Hall, 1997, p. 434f.

[128] Story T.L. Introduction to differential geometry with applications to Navier-Stokes dynamics. New York: iUniverse, 2005, p. 14.

[129] Rodman L. Topics in quaternion linear algebra. Princeton, NJ: Princeton Univ. Press, 2014, pp. 9.36f.

[130] Hand L.N. & Finch J.D. Analytical mechanics. Cambridge: Cambridge Univ. Press, 1998, p. 99ff.

[131] Stover, Christopher. Green's Function. From *MathWorld*--A Wolfram Web Resource, created by Eric W. Weisstein. http://mathworld.wolfram.com/GreensFunction.html

[132] Kirkwood J.R. Mathematical physics with partial differential equations. Waltham, MA: Academic Press, 2013, p. 390ff.

[133] Roach G.F. Green's functions. Cambridge: Cambridge Univ. Press, 1982, pp. 36,2ff.

[134] Hayek S.I. Advanced mathematical methods in science and engineering. New York: Marcel Dekker, 2001, p. 456ff.

[135] Snider A.D. Partial differential equations. Sources and solutions. Mineola, NY: Dover, 2006, p. 42.

[136] Czajko J. Cantor and Generalized Continuum Hypotheses may be false. *Chaos Solit. & Fract.* 21 (2004) 501-512

[137] Euler L. Elements of algebra. New York: Springer, 1984, p. 22f.

[138] Czajko J. On unconventional division by zero. *World Scientific News* 99 (2018) 133-147

[139] Saitoh S. Zero expresses non-possibility. Vixra, 2019.

[140] Dieudonné J. Treatise on analysis II. New York: Academic Press, 1970, p. 151ff.

[141] Michiwaki H., Saitoh S. & Yamada M. Reality of the division by zero z/0=0. *Int. J. Appl. Phys. Math.* 6(1) (2016) 1-8

[142] Michiwaki H., Okumura H. & Saitoh S. Division by Zero z/0 = 0 in Euclidean Spaces. *Int. J. Math. Computation*, 28(1) (2017) 1-16

[143] Okumura H., Saitoh S. & Matsuura T. Relations of 0 and ∞. *JTSS J. Tech. Soc. Sci*. 1(1) (2017) 70-77 see p. 74.

[144] Matsuura T. & Saitoh S. Matrices and division by zero z/0=0. *Adv. Lin. Alg. Matrix Theory* 6 (2016) 51-58

[145] Okumura H. & Saitoh S. Wasan geometry and division by zero calculus. *Sangaku J. Math.* 2 (2018) 57-73

[146] Okumura H. & Saitoh S. The Descartes circles theorem and division by zero calculus. ResearchGate, 2017.

[147] Rogers R. Mathematical logic and formalized theories. Amsterdam: North-Holland, 1974, p. 39.

[148] Needham T. Visual Complex Analysis. Oxford: Clarendon Press, 2001, p.139ff.

[149] Sussman G.J. & Wisdom J. with Farr W. Functional differential geometry. Cambridge, MA: The MIT Press, 2013, p. 19.

[150] Czajko J. On Conjugate Complex Time II: Equipotential Effect of Gravity Retrodicts Differential and Predicts Apparent Anomalous Rotation of the Sun. *Chaos Solit. Fract.* 11 (2000) 2001-2016

[151] Czajko J. Operationally complete work done suggests presence of potentials corresponding to repulsive forces. *Int. Lett. Chem. Phys. Astron.* 37 (2014) 16-30

[152] Czajko J. Angular nonradial vs. usual radial potential energy quotient. *Int. Lett. Chem. Phys. Astron* **52** (2015) 172-177.

[153] Hardy A.S. Elements of quaternions. Boston, 1881, p. 61.

[154] Conway J.H. & Smith D.A. On quaternions and octonions: Their geometry, arithmetic, and symmetry. Wellesley, MA: A.K. Peters, 2003, p. 12.

[155] Wrede R.C. Introduction to vector and tensor analysis. New York: Dover, 1982, p. 3.

[156] Darboux G. Leçons sur la théorie générale des surfaces et les applications géométriques du calcul infinitésimal. Paris: Gauthier-Villars, 1941, p. 458.

[157] De Podesta M. Understanding the properties of matter. London: Taylor & Francis, 2002, p. 23.

[158] Kolata G. Topologists startled by new results. First a famous problem was solved, and now there is proof that there is more than one four-dimensional spacetime. *Science* 217 (1982) 432-433

[159] Saniga M. Quadro quartic Cremona transformations and four-dimensional pencil space-times with the reverse signature. *Chaos Solit. Fract.* 13 (2002) 797-805

[160] Priest G. An introduction of non-classical logic. Cambridge: Cambridge Univ. Press, 2001, p. 64ff.

[161] Hendricks V.F. The convergence of scientific knowledge. A view from the limit. Dordrecht: Kluwer, 2001, p. 92ff.

[162] Klawonn F. Fuzzy points, fuzzy relations and fuzzy functions. [pp. 431-453 in: Novák V. & Perfilieva I. (Eds.) Discovering the world with fuzzy logic. Heidelberg: Physica-Verlag, 2000].

[163] Russell B. The principles of mathematics. London: George Allen & Unwin, 1956, p. 194, pp. 259ff.

[164] Russell B. Introduction to mathematical philosophy. London: George Allen & Unwin, 1970, p. 117ff.

[165] Cusanus. De docta ignorantia. I.11.h 31 [in: Nikolaus von Kues. Philosophisch-theologische Werke. Lateinisch-Deutsch. Darmstadt: Wissenschaftiche Buchgesellschaft.

[166] Singh D. & Singh S.K. Pre-Robinson approach to the theory of infinitesimals. *Int. J. Math. Ed. Sci. Techn*. 23(2) (1992) 249-255

[167] Nicolaus von Kues. Die belehrte Unwissenheit I. Hamburg: Verlag von Felix Meiner, 1979, p.63.

[168] Von Cues N. Von Gottes sehen. Leipzig, 1944, p. 94.

[169] Gminder A. Ebene Geometrie. München, 1932, p. 115.

[170] Dray T. & Manogue C.A. The geometry of the octonions. Singapore: World Scientific, 2015, p. 177.

[171] Girard P.R. Quaternions, Clifford algebras and relativistic physics. Basel: Birkhäuser, 2007, pp. 50,69.

[172] McAulay A. Utility of quaternions in physics. London: Macmillan, 1893, p. 22.

[173] Lemmon W.W. Quaternion invariants of the Minkowski space. New Orleans, LA: Tulane Univ. Tech. Report, 1955, p. 12ff.

[174] McAulay A. Octonions a development of Clifford bi-quaternions. Cambridge: At the Univ. Press, 1898, p. 16ff.

[175] Furey C. Biquaternions and the Clifford algebra CL(2). Cambridge Univ. video 4/14 and subsequent chained videos.

[176] Morais J.P., Georgiev S. & Spröβig W. Real quaternionic calculus handbook. Basel: Springer, 2014, p. 5ff.

[177] Macfarlane A. Vector analysis and quaternions. New York: Wiley, 1906, p. 16ff.

[178] Yano K. Lie derivatives and its applications. Amsterdam: Noordhoff, pp. 9,13ff.

[179] Ślebodziński W. Formes extérieures et leurs applications II. Warszawa: PWN, 1963, p. 192.

[180] Hamilton W.R. Lectures on quaternions: containing a systematic statement of a new mathematical method… Dublin, 1853, see Lecture II § 44.

[181] Born M. Restless universe. New York: Dover, 1951, pp. 73, 139ff.

[182] Witten E. Reflections on the fate of spacetime. *Phys. Today* April 1996, 24-30

[183] Crowell L.B. Quantum fluctuations of spacetime. Singapore: World Scientific, 2006, pp. 243,308ff,339f.

[184] Oprea J. Differential geometry and its applications. Washington, DC: MAA, 2007, pp. 20ff,28ff,30.

[185] Kühnel W. Differential geometry. Curves – surfaces – manifolds. Providence, RI: AMS, 2005, pp. 26ff,30,52,239,348ff.

[186] Stoker J.J. Differential geometry. Delhi: Chaudhury, 2014, p. 63f.

[187] Loudon R. The quantum theory of light. Oxford: Oxford Univ. Press, 2004, p. 129.

[188] Anton H. Calculus. New horizon. New York: Wiley, 1999, p. 850ff.

[189] Ivey T.A. & Landsberg J.M. Cartan for beginners: Differential geometry via moving frames and exterior differential systems. Providence, RI: AMS, 2003, p. 25.

[190] Coxeter H.S.M. Introduction to geometry. New York: Wiley, 1962, p. 321f.

[191] Holm D.D., Schmah T. & Stoica C. Geometric mechanics and symmetry. From finite to infinite dimensions. Oxford: Oxford Univ. Press, 2009, p. 140.

[192] Coburn N. Vector and tensor analysis. New York: Dover, 1970, p. 64, p. 317ff.

[193] Holm D.D. Geometric mechanics part II: Rotating, translating and rolling. Singapore: World Scientific, 2008, p. 66.

[194] Pennisi L.L., Gordon L.I. & Lasher S. Elements of complex variables. New York: Holt, Rinehart and Winston, 1963, pp. 90,305f,308.

[195] Penrose R. The road to reality. A complete guide to the laws of the universe. New York: Alfred A. Knopf, 2005, pp. 500ff,505f.

[196] Thaller B. Visual quantum mechanics. New York: Springer, 2000, pp. 27,34,51ff,57.

[197] Ferrero M. & Santos E. The wave-particle duality and the Aharonov-Bohm effect. [pp. 29-36 in: Van der Merwe A. & Garuccio A. (Eds.) Waves and particles of light and matter. New York: Plenum Press, 1994, see p. 31].

[198] Mugur-Schächter M. p=h/λ? Whv? A riddle prior to any attempt at grand unification. [pp. 541-569 in: Van der Merwe A. & Garuccio A. (Eds.) Waves and particles of light and matter. New York: Plenum Press, 1994, see p. 542f].

[199] Hendricks V.F. The convergence of scientific knowledge. A view from the limit. Dordrecht: Kluwer, 2001, p. 83.

[200] Kroemer H. Quantum mechanics for engineers, materials science and applied physics. Englewood Cliffs, NJ: Prentice-Hall, 1994, p. 4.

[201] Bohm D. Causality and chance in modern physics. Princeton, NJ: Van Nostrand, 1957, pp. 83,84.

[202] De Broglie L. Matter and light. The new physics. New York: Dover, [no date], p. 267.

[203] Longair M.S. Theoretical concepts in physics. An alternative view of theoretical reasoning in physics. Cambridge: Cambridge Univ. Press, 2006, pp. 421, 422.

[204] Hadamard J. Four lectures on mathematics. New York: Columbia Univ. Press, 1915, pp. 4,5,8f,10.

[205] Shatah J. & Struwe M. Geometric wave equations. New York: Courant Institute, 1998, p. 82ff.

[206] Friedlander F.G. The wave equation on a curved space-time. Cambridge: Cambridge Univ. Press, 1975, p. 178ff.

[207] Synge J.L. Geometrical mechanics and De Broglie waves. Cambridge: Cambridge Univ. Press, 1954, pp. 64,105.

[208] Willmore T.J. An introduction to differential geometry. Mineola, NY: Dover, 2012, p. 137.

[209] Lanczos C. Linear differential operators. Mansfield Centre, CT: Martino Publishing, 2012, p. 336.

[210] Rindler W. Introduction to special relativity. Oxford: Clarendon Press, 1991, pp. 17,27.

[211] Harpaz A. Relativity theory. Concepts and basic principles. Boston: Jones and Bartlett, 1992, p. 60ff.

[212] Kleinerman S. Global existence for nonlinear wave equations. *Commun. Pure Appl. Math.* 33 (1980) 43-101

[213] John F. Nonexistence of global solutions of $\Box u = F(u_{tt})$ and of $\Box v = F'(v_t)v_{tt}$ in three space dimensions. *Proc. Nat. Acad. Sci. USA* 77 (1980) 1759-1760.

[214] Hurley P. Time in the earlier and later Whitehead. [pp. 87-109 in: Griffin D.R. (Ed.) Physics and the ultimate significance of time. Albany, NY: State Univ. of New York Press, 1986].

[215] Cole E.A.B. Emission and absorption of tachyons in six-dimensional relativity. *Phys. Lett*. 75A (1979) 29-30.

[216] Cole E.A.B. Superluminal transformations using either complex space-time or real space-time symmetry. *N. Cim*. 40A(2) (1977) 171-180

[217] Henderson D.W. Differential geometry. A geometric introduction. Upper Saddle River, NJ: Prentice-Hall, 1998, p. 60.

[218] Weisstein, Eric W. Darboux Vector. From *MathWorld*--A Wolfram Web Resource. http://mathworld.wolfram.com/DarbouxVector.html

[219] Ertel H. Ein neuer hydrodynamischer Wirbelsatz. *Met. Z.* 59 (1942) 271-281

[220] Gibbon J.D. & Holm D.D. Lagrangian particle paths & ortho-normal quaternion frames. arXiv, 2007.

[221] Leithold L. The calculus with analytic geometry. Part II: Infinite series, vectors, and functions of several variables. New York: Harper & Row, Publishers, 1972, p. 791.

[222] Stein F.M. An introduction to vector analysis. New York: Harper & Row, 1963, p. 43f.

[223] Faber R.L. Differential geometry and relativity theory. An introduction. New York: Marcel Dekker, 1983, p. 146.

[224] Moon P. & Spencer D.E. Field theory handbook including coordinate systems, differential equations and their solutions. Berlin: Springer, 1971, pp. 3,136f.

[225] Weisstein, Eric W. Vector Laplacian. From *MathWorld*--A Wolfram Web Resource. http://mathworld.wolfram.com/VectorLaplacian.html

[226] Misner C.W., Thorne K.S. & Wheeler J.A. Gravitation. New York: Freeman, 1973, pp. 393,1055.

[227] Czajko J. On the Hafele-Keating Experiment. *Ann. Physik (Leipzig)* 47 (1990) 517-518

[228] French A.P. Special relativity. New York: W.W. Norton, 1968, p. 25f.

[229] Lorentz H.A. The theory of electrons and its applications to the phenomena of light and radiant heat. New York: The Columbia Univ. Press, 1909, p. 225.

[230] Newton T.D. & Wigner E.P. Localized states for elementaary systems. *Rev. Mod. Phys*. 21(3) (1949) 400-406

[231] Fleming G.N. Nonlocal properties of stable particles. *Phys. Rev*. B 139 (1965) 963

[232] Ruijsenaars S.N.M. On Newton-Wigner localization and superluminal propagation speeds. *Ann. Phys*. 137 (1981) 33-43

[233] Silagadze Z.K. The Newton-Wigner position operator and the domain of validity of one-particle relativistic theory. CERN: *SLAC Publ*. (1993).

[234] Westra W. Localization of particles in quantum field theory. arXiv, 2010, see p.5f.

[235] Shirokov M.I. Packet spreading and Einstein retardation. *Concepts Phys*. 6(4) (2009) 543-552 and see references therein

[236] Hagerfeldt G.C. Comment on Packet Spreading and Einstein retardation. *Concepts Phys.* 6(4) (2009) 553-554.

[237] Thom R. To predict is not to explain. Thombooks Press, 2015, p. 85.

[238]] Saitoh S. Division by zero functions in trigonometric functions. Vixra, 2019.

[239] Czajko J. On Conjugate Complex Time I: Complex Time Implies Existence of Tangential Potential that Can Cause Some Equipotential Effects of Gravity. *Chaos, Solit. Fractals* 11 (2000) 1983-1992

[240] Czajko J. Experiments with flying atomic clocks. *Exper. Techn. Phys*. 39 (1991) 145-147

[241] Czajko J. Operationally complete work done suggests presence of potentials corresponding to repulsive forces. *Int. Lett. Chem. Phys. Astron.* 37 (2014) 16-30