ZESZYTY PROBLEMOWE POSTĘPÓW NAUK ROLNICZYCH 1976, z. 168

CLASSIFICATION OF RHEOLOGICAL MODELS COMPOSED OF SPRINGS AND DASHPOTS

Zoltán Müller

University of Agricultural Sciences, Agricultural Engineering Faculty, Gödöllö

INTRODUCTION

Planning of tilth and designing of tillage tools require such a thorough knowledge of the mechanical properties of agricultural soils as possible. In course of tilth the equilibrum existing earlier in the soil breaks down, a new distribution of stress develops followed by a timedependent deformation and a transient effect in stress and the process settles only after a space of time. Thus, certainly it is necessary to be acquainted with the changes in the mechanical properties (deformation, stress) of the soil for a longer period. The course of the process also depends on the mechanical history of the soil, which must be taken into account by investigating any new process. Therefore, it is not sufficient to investigate merely the mechanical parameters (deformation, stress) of the soil, but also their evolution in time is to be taken into consideration and such a model is to be developed for the soil under investigation, that is capable to describe the behaviour of the soil in any period of time.

MODELLING OF SOILS

In course of the macrorheological investigation of soils a one-to-one phenomenological relationship is ought for between two fundamental dynamical and kinetical quantities: the stress (force, moment) and the relative deformation (tensile or angular strain, distortion). This can be done most naturally by considering the specimen as a system, and investigating how do the output parameters of the system change as functions of time, if the input parameters vary in time regularly. The specimen, as a system, is illustrated by the scheme on Fig. 1. where $\overline{X} = \overline{X}(t)$ is the vector of input parameters, $\overline{Y} = \overline{Y}\,(t)$ is the vector of output parameters.

In general, the system can be described only by simultaneous relationships between several input and output parameters. In recent case we restrict ourselves on systems with one variable (one input and one output variable), such as uniaxial compression. In case of uniaxial rheological experiments the phenomenon can be described by the equation of state

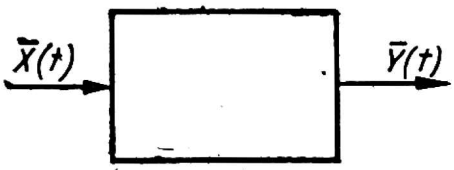


Fig. 1. Soil sample as a system

$$f\,(\sigma, \varepsilon, t) = 0. \tag{1}$$

Rheological phenomena can generally be divided into two classes: creeping phenomena and relaxation phenomena. In case of creeping phenomena the generator function $\sigma\ (t)$ is regarded as input parameter, the response function $\varepsilon\ (t)$ as output parameter. Inversely, in case of relexation phenomena $\varepsilon\ (t)$ is taken as input, $\sigma\ (t)$ as output. Here $\sigma\ (t) = \dfrac{E(t)}{A_0}$ is the nominal stress, $\varepsilon\ (t) = \dfrac{\Delta l(t)}{l_0}$ is the relative deformation, both regarded as functions of time.

The specimen prepared from agricultural soil can generally be regarded as a continuum exhibiting elastic and viscoelastic properties. With respect to its rheological behaviour, in many cases this continuum can be approximated well by a model composed of concentrated linear elements, springs and dashpots. The mass plays no role in this case, inasmuch as the processes take place slowly and the originating inertial forces can be neglected. Since the model contains only linear elements and the elements of the system are considered time invariant, the model can be described by a linear differential equation with constant coefficients. The general form of the differential equation is

$$\sigma + \sum_{k=1}^{n} b_k \frac{d^k\sigma}{dt^k}\ a_0\varepsilon + \sum_{k=1}^{m} a^k \frac{d^k\varepsilon}{dt^k}\,. \tag{2}$$

Here, the relation between m and n can be either m=n or m=n+1. In certain cases the value of the constant a_0 is $a_0 = \Theta$. The general solution of differential equation (2) is composed of two parts:

— the general or transient solution of the homogeneous equation and,

— the particular or stationary solution of the inhomogeneous equation.

The general solution of differential equation (2) satisfying the initial conditions yields the response function of the model for a given input signal.

The Laplace transformation is an excellent tool for solving differential equation (2), respectively for determining the response function, since it converts the differential equation and the response function into algebraic equation. If the system was energy-free before the effect of the input signal, then we obtain

$$\frac{\sigma(p)}{\varepsilon(p)} = Z^x(p),$$

as the Laplace transform of differential equation (2), where

$$Z^x(p (= \frac{a_0 + pa_1 + p^2a_2 + \cdots\cdots + p^m a_m}{1 + pb_1 + p^2b_2 + \cdots\cdot p^n b_n} . \tag{4}$$

Here p is a complex variable, its dimension is [1/s]. The possible values of m and n are again either man or $m = n + 1$.

The fuction Z^x (p) relating generation and response depends only on the properties of the system (specimen), that is to say, it is a characteristic fuction of the sytem termed also transfer function. It can be concluded from equation (4) that the transfer function can be characterized by $(m + n + 1)$ data. Coefficients a and b of function Z^x (p) are always real numbers.

The transfer function Z^x (p) of equation (4) can be transcribed by root-factorization into the form

$$Z^x(p) = \frac{a_m(p - s_1)(p - s_2)\cdots(p - s_m)}{b_n(p - p_1)(p - p_2)\cdots(p - p_n)} . \tag{5}$$

In equation (5) the p_i's mean the root loci of the denominator, that is, the poles of Z^x (p) (denoted with x), while the s_i's mean the root loci of the numerator, that is the zeros of Z^x (p) (denoted with O). Thus by the ratio a_m/b_n together with the pattern of poles and zeros the model can be completely characterized (see Figures 7-16). The poles and zeros of the transmission function Z^x (p) of models composed of springs and dashpots, if plotted on the complex plane, are arranged alternately on the negative real axis.

CONNECTION OF SPRINGS AND DASHPOTS

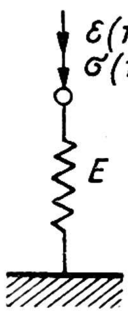
Being acquainted with the transfer function Z^x (p) of a model the connection of the model composed of springs and dashpots as well as the values of its elements can be determined. The number of constants

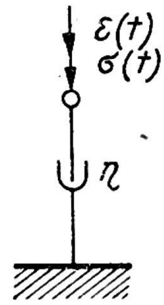
a and b in the function Z^x (p) equals to the minimal number of springs and dashpots constituting the model. The connection of the model can be determined most simply by decomposing the function Z^x (p) into partial fractions (Foster synthesis) or expanding it into chain fractions (Cauer synthesis). By use of various procedures several models different in connection but completely equivalent in behaviour can be obtained (see Figures 7-16). The constants of models of different connection but equivalent behaviour can be converted in each other on the basis of their Z^x (p) function.

According to equations (2), (3), and (4) the Z^x (p) function of a model composed of a single spring is

$$Z^x(p) = E,$$

E is the elastic modulus [kp/m²]. The model is shown in Fig. 2.



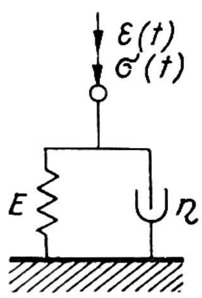


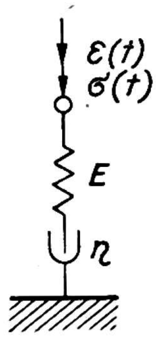
Figs. 2 and 3. Models consisting of one spring and one dashpot respectively

The Z^x (p) function of a model composed of a single dashpot is

$$Z^x(p) = p\,\eta,$$

η is the viscosity [kp · s/m²] The model can be viewed in Fig. 3.





Figs. 4 and 5. Different connections of spring and dashpot

Figure 4 exhibits the scheme of a model consisting of the parallel array of a spring and a dashpot. The Z^x (p) function of the model is

$$Z^x(p) = E + p\,\eta.$$

In case of series array the inverse summation of the Z^x (p) functions of the spring and the dashpot yields the inverse of the resultant Z^x (p) function

$$Z^x(p) = E \times p\eta = \frac{pE\eta}{E + p\eta}.$$

The model of series array is illustrated in Fig. 5.

The compound connection of springs and dashpots is demonstrated in Fig. 6. The parallel array of the spring E_2 and the dashpot η_2 is connected in series with spring E_1. The resultant Z^x (p) function is

$$Z^x(p) = E_1 \times (E_2 + p\,\eta_2),$$

Symbol $\times$ refers to inverse summation.

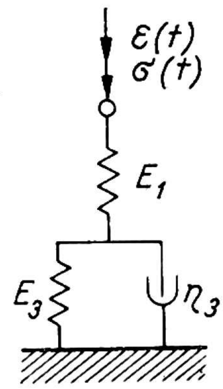
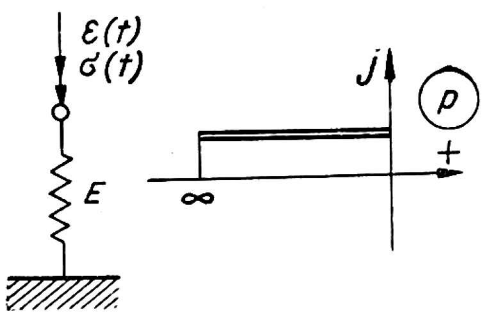


Fig. 6. Complex rheologic model



Figs. 7-16. Several models different in connections but equivalent in behaviour

On the above bases the Z^x (p) function of any model composed of springs and dashpots can easily be constructed.

CLASSIFICATION OF MODELS COMPOSED OF SPRINGS AND DASHPOTS

Rheological models can also be classified according to the value of their Z^x (p) function at $p = 0$ and $p = \infty$. This method of classification provides a rather good survey on models composed of springs and dashpots connected in various ways. Namely, as it could be seen before, models of different connection may possess the same Z^x (p) function, thus the types of models can be characterized best by their Z^x (p) functions and their plot on the p-plane. It can be concluded from equation (4) that at $p = 0$ the function Z^x (p) may assume two sort of values depending on whether $a_o = 0$ or $a_o \neq 0$, namely:

$$p = 0 \begin{cases} Z^x(p) = \text{real finite} & (a_o \neq 0), \\ Z^x(p) = 0 & (a_o = 0). \end{cases}$$

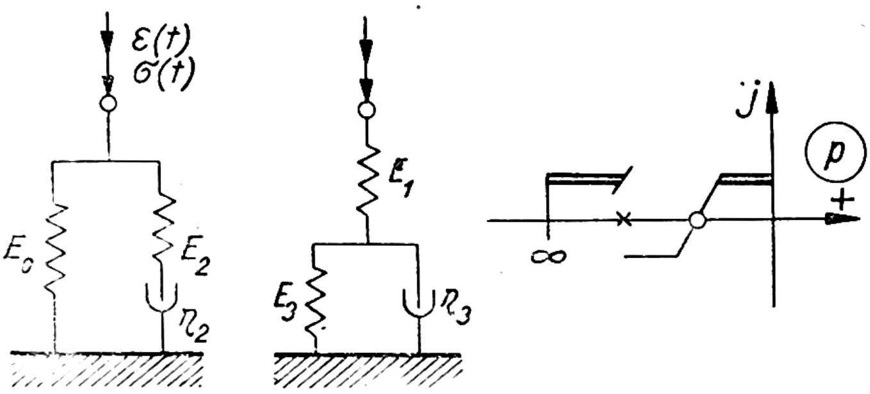


Fig. 8.

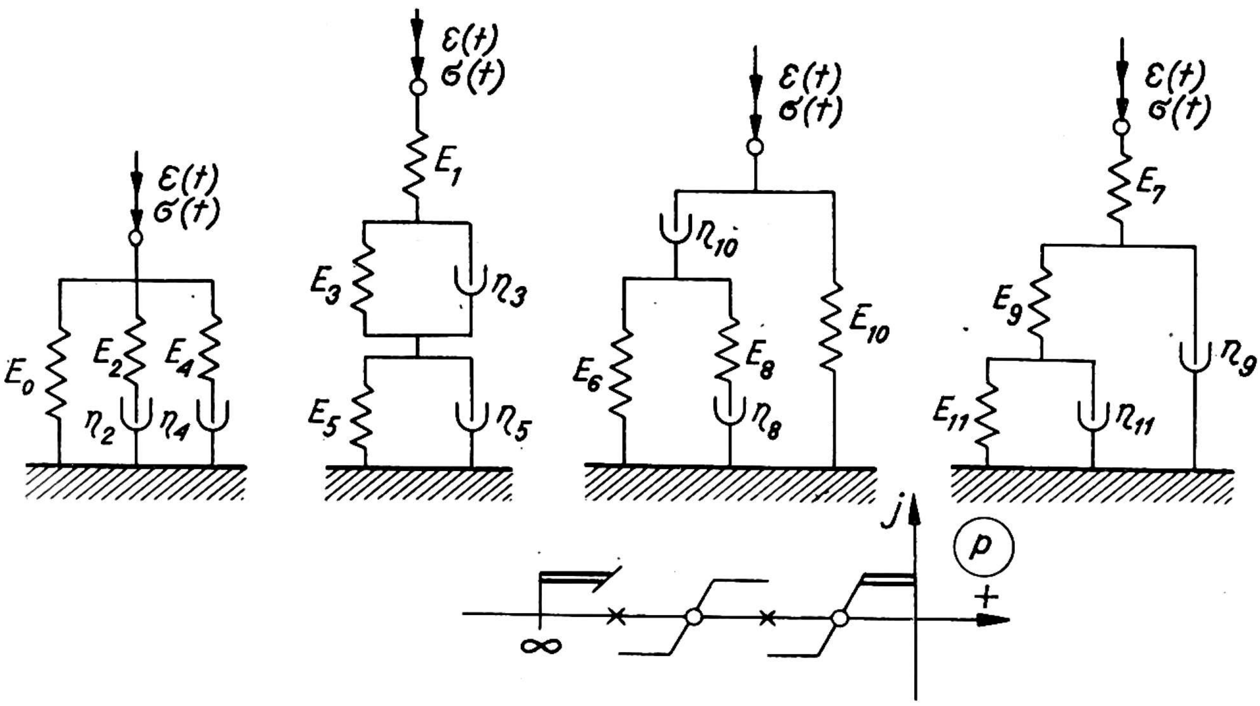


Fig. 9.

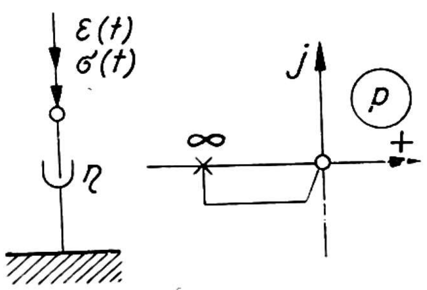


Fig. 10.

At $p = \infty$ Z^x (p) may assume again two sort of values depending on the relation between the degree of the denominator and the numerator

$$p = \infty \begin{cases} Z^x(p) = \text{real finite} & (m = n), \\ Z^x(p) = \infty & (m = n + 1). \end{cases}$$

On that basis all of the models composed of springs and dashpots may be divided into four classes according to Table 1.

In Table 2 the differential equations and Z^x (p) functions are indicated for every class. The connections of the models as well as their symbolical scheme on the p-plane are shown in Figures 7-16.

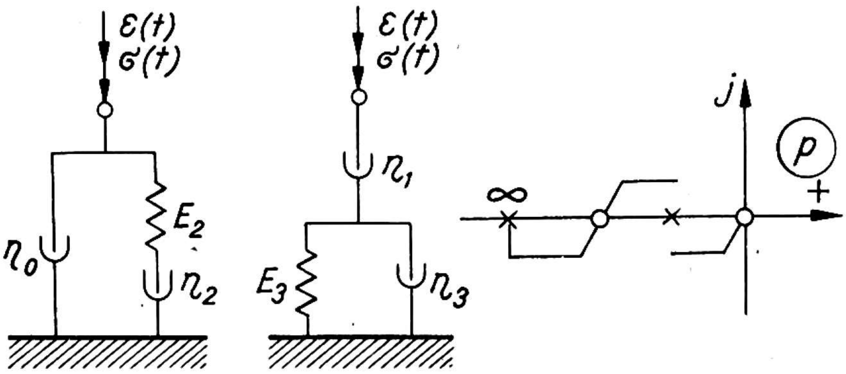


Fig. 11.

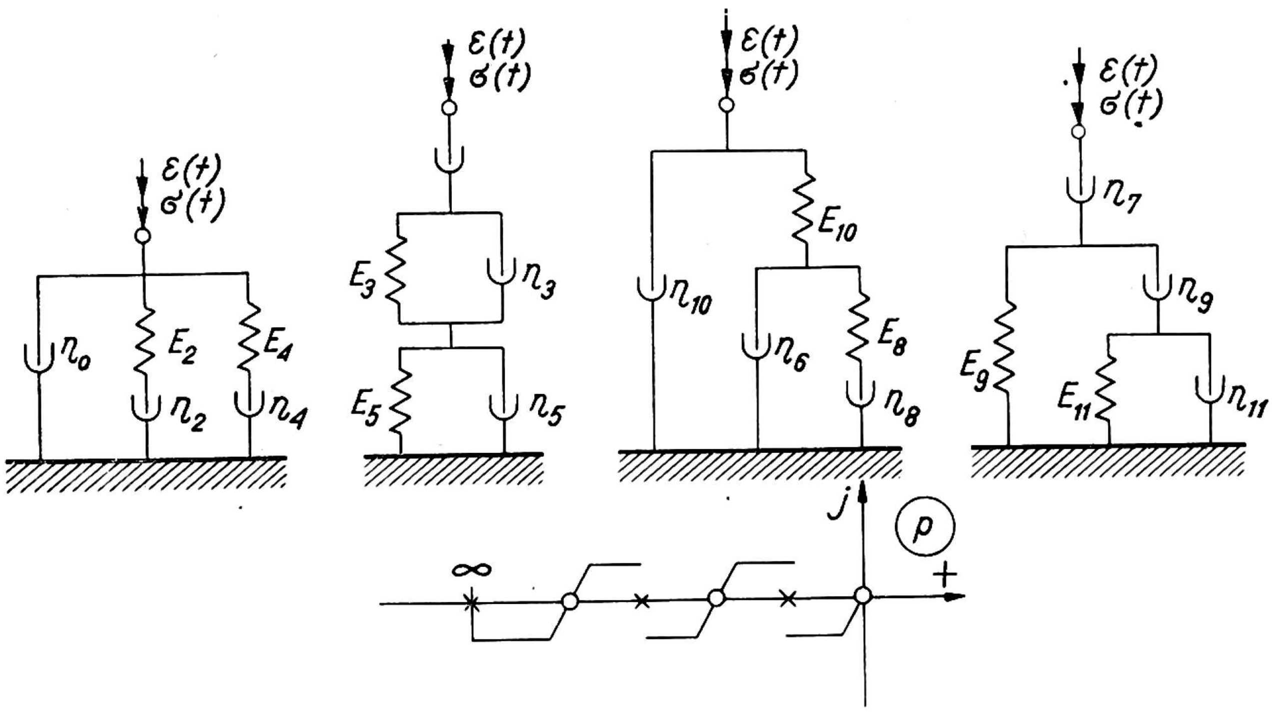


Fig. 12.

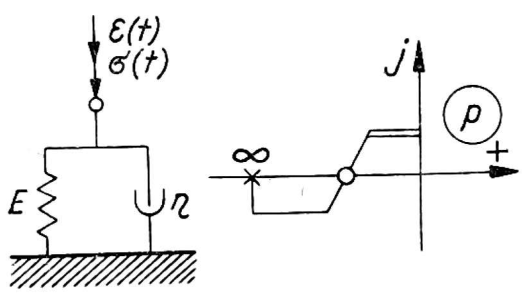


Fig. 13.

Table 2 gives a guidance concerning to the possibility of extension of ted for every class. The connections of the models as well as their sym- transfer functions Z^x (p).

On the basis of our measurements the rheological behaviour of agricultural soils proved to be best approximated by the models of classes I and III.

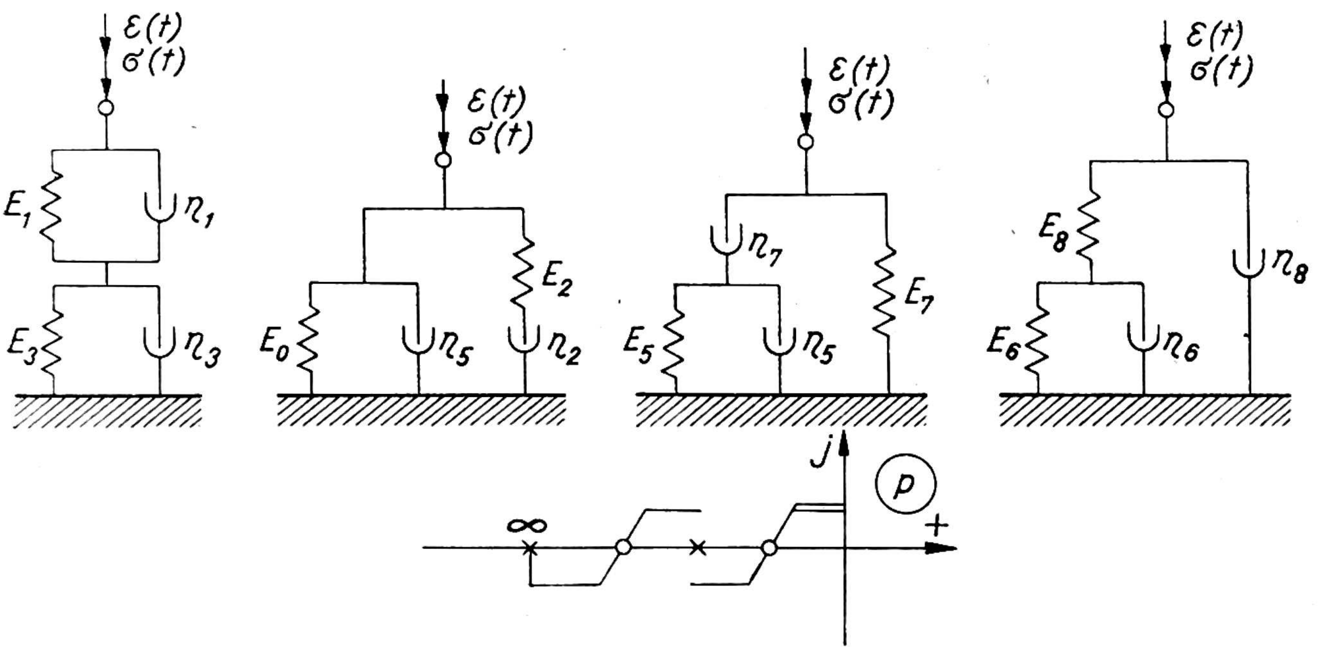


Fig. 14.

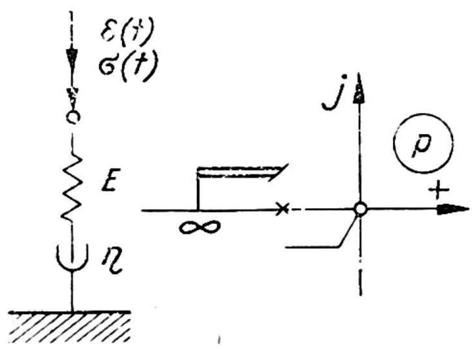


Fig. 15.

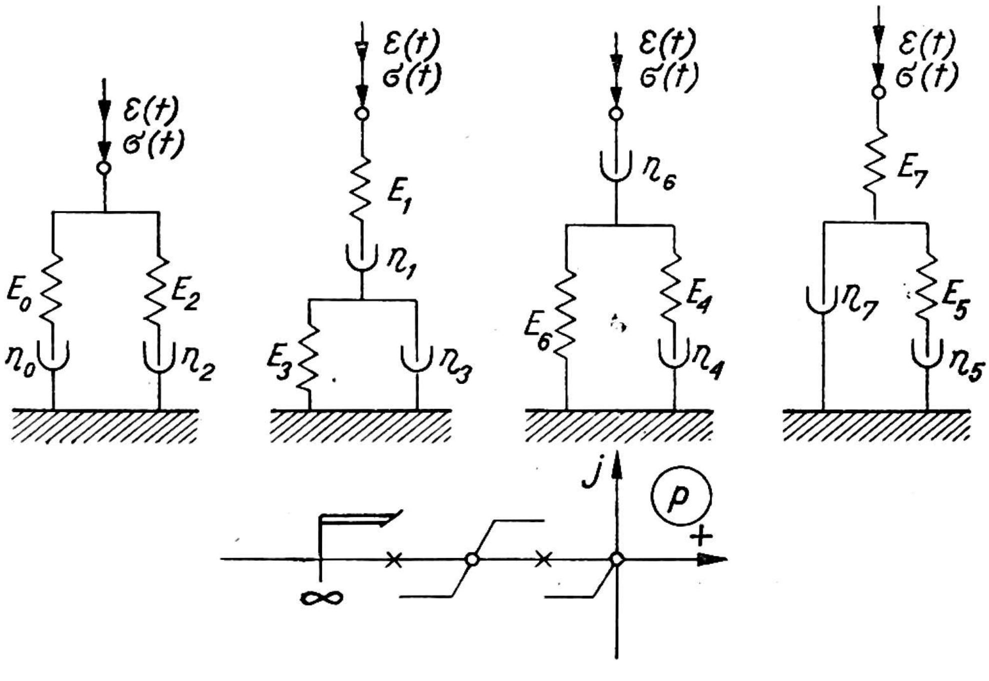


Fig. 16.

Table 1

Classification of models

Class	$Z^x(p)$, $p = 0$	$Z^x(p)$, $p = \infty$	Comment
I	real finite	real finite	Hooke, Poynting—Thomson etc.
II	0	∞	Newton, Jeffreys etc.
III	real finite	∞	Voigt—Kelvin etc.
IV	0	real finite	Maxwell, Burgers, etc.

Table 2

Differential equations and $Z^x(p)$ functions of models

Name	Figure	Differential equation	Transfer function $Z^x(p)$
Class I			
Hooke elastic body	7	$\sigma = a_0 \varepsilon$	$\frac{a_0}{1}$
Poyting-Thomson body	8	$\sigma = a_0 \varepsilon + a_1 \dot{\varepsilon} - b_1 \dot{\sigma}$	$\frac{a_0 + pa_1}{1 + pb_1}$
Extended Poyting-Thomson body (5 terms)	9	$\sigma = a_0 \varepsilon + a_1 \dot{\varepsilon} - b_1 \dot{\sigma} + a_2 \ddot{\varepsilon} - b_2 \ddot{\sigma}$	$\frac{a_0 + pa_1 + p^2 a_2}{1 + pb_1 + p^2 b_2}$
Class II			
Newton fluid body	10	$\sigma = a_1 \dot{\varepsilon}$	$\frac{pa_1}{1}$
Jeffreys model	11	$\sigma = a_1 \dot{\varepsilon} - b_1 \dot{\sigma} + a_2 \ddot{\varepsilon}$	$\frac{pa_1 + p^2 a_2}{1 + pb_1}$
Extended Jeffreys model (5 terms)	12	$\sigma = a_1 \dot{\varepsilon} - b_1 \dot{\sigma} + a_2 \ddot{\varepsilon} - b_2 \ddot{\sigma} + a_3 \dddot{\varepsilon}$	$\frac{pa_1 + p^2 a_2 + p^3 a_3}{1 + pb_1 + p^2 b_2}$
Class III			
Voigt-Kelvin body	13	$\sigma = a_0 \varepsilon + a_1 \dot{\varepsilon}$	$\frac{a_0 + pa_1}{1}$
Extended Voigt-Kelvin body (4 terms)	14	$\sigma = a_0 \varepsilon + a_1 \dot{\varepsilon} - b_1 \dot{\sigma} + a_2 \ddot{\varepsilon}$	$\frac{a_0 + pa_1 + p^2 a_2}{1 + pb_1}$
Class IV			
Maxwell body	15	$\sigma = a_1 \dot{\varepsilon} - b_1 \dot{\sigma}$	$\frac{pa_1}{1 + pb_1}$
Burgers body	16	$\sigma = a_1 \dot{\varepsilon} - b_1 \dot{\sigma} + a_2 \ddot{\varepsilon} - b_2 \ddot{\sigma}$	$\frac{pa_1 + p^2 a_2}{1 + pb_1 + p^2 b_2}$

REFERENCES

1. Fodor G.: Analysis of Linear Systems, Budapest, 1967.
2. Géher K.: Linear Networks, I., II., Budapest, 1963.
3. Mózes G., Vámos E.: Rheology and Rheometry, Budapest, 1968.
4. Asszonoyi C., Huszár J., Kapolyi: Laboratory determination of rock constants by creeping test, Tatabányai Szénbányák Müszaki-Közgazdasági Közleményei, 1° (4), 177-182, 1971.

Z. Müller

KLASYFIKACJA MODELI REOLOGICZNYCH SKŁADAJĄCYCH SIĘ ZE SPRĘŻYN I TŁUMIKÓW

Streszczenie

Autor wykazał, że reologiczne zachowanie się wielu rodzajów gleb może być z wystarczającym przybliżeniem opisane za pomocą modeli reologicznych składających się ze sprężyn i tłumików. Modele te we wszystkich przypadkach dają się sprowadzić do liniowego równania różniczkowego o stałym współczynniku. Przy pomocy transformacji Laplace'a można podzielić wszystkie tego typu modele na cztery klasy. Każdą klasę charakteryzuje podobieństwo zachowania się modeli i równań opisujących je.

З. Мюллер

КЛАССИФИКАЦИЯ РЕОЛОГИЧЕСКИХ МОДЕЛЕЙ СОСТАВЛЕННЫХ ИЗ ПРУЖИН И БУФЕРОВ

Резюме

Автор установил, что реологическое поведение многих видов почв можно с удовлетворительным приближением описать с помощью реологических моделей, составленных из пружин и буферов. Эти модели можно во всех случаях свести к линейному дифференцияльному уравнению с постоянным коэффициентом. С помощью трансформации Лапласа все этого рода модели можно разделить на четыре класса. Каждый класс характеризуется сходным поведением моделей и описывающих их уравнений.